Chapter 5

Perturbation Theory

In this chapter we will discuss time dependent perturbation theory in classical mechanics. Many problems we have encountered yield equations of motion that cannot be solved analytically. Here, we will consider cases where the problem we want to solve with Hamiltonian $H(q,p,t)$ is "close" to a problem with Hamiltonian $H_0(q,p,t)$ for which we know the exact solution. We say

$$H(q,p,t) = H_0(q,p,t) + \Delta H(q,p,t)\,, \tag{5.1}$$

where ΔH is small. The general idea is to expand variables

$$z(t) = z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + \ldots\,, \tag{5.2}$$

for $z \in \{q,p\}$ and use the expanded equations of motion to determine the series

$$z_0(t) \to z^{(1)}(t) \to z^{(2)}(t) \to \ldots\,, \quad \text{where} \quad z^{(k)}(t) = \sum_{j=0}^{k} \epsilon^j z_j(t)\,. \tag{5.3}$$

We can do this with any of our methods for solving problems in classical mechanics, including the Euler-Lagrange equations, Hamilton equations, Poisson bracket equations, or Hamilton-Jacobi equations. Since there are some practical benefits, our focus will be on doing this for the Hamilton-Jacobi equations, but lets first start with an example where we carry out an expansion for the Hamilton equations.

Example Consider $H_0 = \frac{p^2}{2m}$ the free Hamiltonian, and $\Delta H = \frac{m\omega^2}{2}x^2$. Here ω is an oscillator frequency. The full Hamiltonian $H = H_0 + \Delta H$ in this case is just a Harmonic oscillator where we already know the solution, so we have the opportunity to see how this solution is built up perturbatively. Without any approximation, the Hamilton equations are

$$\dot{x} = \frac{p}{m}\,, \qquad\qquad \dot{p} = -m\omega^2 x\,. \tag{5.4}$$

To carry out perturbation theory we are going to count w^2 as $\mathcal{O}(\epsilon)$ and then at each order we balance the number of ϵs on each side of the equations of motion. For H_0, we have

$\dot{p}_0 = 0$ so the momentum p_0 is a constant which we fix as the initial condition value. We also have $\dot{x}_0 = \frac{p_0}{m}$ (since w^2 does not appear we have not dropped anything in this equation). Integrating we get $x_0 = \frac{p_0}{m}t$, where we have taken the initial condition $x(t=0) = 0$ for simplicity.

Having setup the 0'th order solution, lets now consider determining the solution at 1'st order. At first order the RHS of the equations of motion should be $\mathcal{O}(\epsilon)$. Therefore

$$\begin{aligned} \dot{p}^{(1)} &= -m\omega^2 x^{(0)} = -m\omega^2 \frac{p_0}{m}\, t = -\omega^2 t\, p_0, \\ p^{(1)}(t) &= p_0 - \frac{1}{2} p_0 \omega^2 t^2. \end{aligned} \tag{5.5}$$

For the other equation of motion at this order we then have

$$\begin{aligned} \dot{x}^{(1)} &= \frac{p^{(1)}}{m} = \frac{p_0}{m} - \frac{p_0\omega^2 t^2}{2m}, \\ x^{(1)}(t) &= \frac{p_0}{m} t - \frac{p_0 \omega^2 t^3}{6m}. \end{aligned} \tag{5.6}$$

These are precisely the 1^{st} order terms in the full solution

$$p(t) = p_0 \cos(\omega t), \qquad x(t) = \frac{p_0}{m\omega} \sin(\omega t). \tag{5.7}$$

5.1 Time Dependent Perturbation Theory for the Hamilton-Jacobi Equations

From here on we will focus on using H-J methods. If $H = H_0 + \Delta H$, then the solution for H_0 has a principal function $S(q, \alpha, t)$ that is the generating function that makes a canonical transformation $(q, p) \to (\alpha, \beta)$, so that

$$H_0\Big(q, \frac{\partial S}{\partial q}, t\Big) + \frac{\partial S}{\partial t} = 0\,. \tag{5.8}$$

For the dynamics generated by H_0 the variables (α, β) are constants. However, the resulting canonical transformation provides a new set of variables that is valid for use with any Hamiltonian, they are just particularly simple variables for H_0. Therefore, for H, we can still use the canonical transformation generated by S, but now the new variables

$$P_i = \alpha_i = \alpha_i(p, q)\,, \qquad\qquad Q_i = \beta_i = \beta_i(p, q)\,, \tag{5.9}$$

will no longer be constant in time. The new Hamiltonian is

$$K = H_0 + \Delta H + \frac{\partial S}{\partial t} = \Delta H = \Delta H(\alpha, \beta, t) \tag{5.10}$$

The new Hamilton equations $\dot{Q}_i = \frac{\partial K}{\partial P_i}$ and $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$ now yield exact equations of motion for these variables

$$\dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i}\,, \qquad\qquad \dot{\beta}_i = \frac{\partial \Delta H}{\partial \alpha_i}\,. \tag{5.11}$$

The idea of perturbation theory is to solve these equations with an expansion. Since here the small $\Delta H \sim \epsilon$ appears on the RHS of both equations, we will always use lower order solutions on the RHS to obtain the higher order results on the LHS. Thus we use $\alpha^{(0)}$ and $\beta^{(0)}$ to get the first order $\alpha^{(1)}$ and $\beta^{(1)}$:

$$\begin{aligned}
\dot{\alpha}_i^{(1)} &= -\left.\frac{\partial \Delta H}{\partial \beta_i}\right|_{\substack{\alpha_i = \alpha_i^{(0)} \\ \beta_i = \beta_i^{(0)}}} = -\left.\frac{\partial \Delta H}{\partial \beta_i}\right|_0\,, \\
\dot{\beta}_i^{(1)} &= \left.\frac{\partial \Delta H}{\partial \alpha_i}\right|_{\substack{\alpha_i = \alpha_i^{(0)} \\ \beta_i = \beta_i^{(0)}}} = \left.\frac{\partial \Delta H}{\partial \alpha_i}\right|_0\,,
\end{aligned} \tag{5.12}$$

where the $|_0$ is a shorthand notation. We then use $\alpha^{(1)}$ and $\beta^{(1)}$ to get $\alpha^{(2)}$ and $\beta^{(2)}$, and so on. At n$^{\text{th}}$ order we have

$$\begin{aligned}
\dot{\alpha}_i^{(n)} &= -\left.\frac{\partial \Delta H}{\partial \beta_i}\right|_{\substack{\alpha_i = \alpha_i^{(n-1)} \\ \beta_i = \beta_i^{(n-1)}}} = -\left.\frac{\partial \Delta H}{\partial \beta_i}\right|_{n-1}\,, \\
\dot{\beta}_i^{(n)} &= \left.\frac{\partial \Delta H}{\partial \alpha_i}\right|_{\substack{\alpha_i = \alpha_i^{(n-1)} \\ \beta_i = \beta_i^{(n-1)}}} = \left.\frac{\partial \Delta H}{\partial \alpha_i}\right|_{n-1}\,.
\end{aligned} \tag{5.13}$$

Example Lets once again consider $H_0 = \frac{p^2}{2m}$ and $\Delta H = \frac{m\omega^2}{2}x^2$. For H_0, the H-J equation is $\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t} = 0$. As x is cyclic, the solution is $S = \alpha x - \frac{\alpha^2}{2m}t$. Here,

$$P = \alpha\,, \qquad\qquad Q = \beta = \frac{\partial S}{\partial \alpha} = x - \frac{\alpha}{m}t\,, \tag{5.14}$$

giving the exact transformation equations

$$x = \frac{\alpha}{m}t + \beta\,, \qquad\qquad p = \frac{\partial S}{\partial x} = \alpha\,.$$

For simplicity, we can take the initial constants as $\alpha^{(0)} = \alpha_0$ and $\beta^{(0)} = \beta_0 = 0$. In terms of the new variables our perturbing Hamiltonian is $\Delta H = \frac{m\omega^2}{2}\left(\frac{\alpha}{m}t + \beta\right)^2$, so prior to expanding the full equations of motion are

$$\begin{aligned} \dot{\alpha} &= -\frac{\partial \Delta H}{\partial \beta} = -m\omega^2\left(\frac{\alpha}{m}t + \beta\right)\,, \\ \dot{\beta} &= \frac{\partial \Delta H}{\partial \alpha} = \omega^2 t\left(\frac{\alpha}{m}t + \beta\right)\,. \end{aligned} \tag{5.15}$$

Plugging in 0^{th} order solutions on the RHS, to 1^{st} order we have

$$\begin{aligned} \dot{\alpha}^{(1)} &= -\omega^2\alpha_0 t \quad \Rightarrow \quad \alpha^{(1)}(t) = \alpha_0 - \frac{1}{2}\omega^2\alpha_0 t^2\,, \\ \dot{\beta}^{(1)} &= \frac{\omega^2}{m}\alpha_0 t^2 \quad \Rightarrow \quad \beta^{(1)}(t) = \frac{\omega^2\alpha_0 t^3}{3m}\,. \end{aligned} \tag{5.16}$$

If we change back to our original variables with the inverse transformation (which we may wish to do at any point) this gives

$$p^{(1)} = \alpha^{(1)} = \alpha_0 - \frac{1}{2}\omega^2\alpha_0 t^2, \tag{5.17}$$

and

$$x^{(1)}(t) = \frac{\alpha^{(1)}(t)}{m}t + \beta^{(1)}(t) = \frac{\alpha_0}{m}t - \frac{\omega^2\alpha_0}{m}\frac{t^3}{2} + \frac{\omega^2\alpha_0}{m}\frac{t^3}{3} = \frac{\alpha_0}{m}t - \frac{\omega^2\alpha_0}{m}\frac{t^3}{3!}\,, \tag{5.18}$$

which are the same results we previously obtained by solving Hamilton's equations perturbatively.

5.2 Periodic and Secular Perturbations to Finite Angle Pendulum

Example Let us consider a case where we do not have a simple solution. Consider a pendulum, with

$$H = \frac{p^2}{2ma^2} - mga\cos(\theta) \tag{5.19}$$

with $\theta \ll 1$. Expanding the cosine term we have

$$H = -mga + \frac{p^2}{2ma^2} + \frac{mga}{2}\theta^2 + \frac{mga}{2}\theta^2\left(-\frac{\theta^2}{12} + \frac{\theta^4}{360} + \dots\right). \tag{5.20}$$

In this case, the first term is a constant that will not play a role in our equations of motion, so we can identify $H_0 = \frac{p^2}{2ma^2} + \frac{mga}{2}\theta^2$. If we are only interested in applying first order perturbation theory we can simply take $\Delta H = -\frac{mga}{24}\theta^4$ and drop terms of $\mathcal{O}(\theta^6)$ and higher. The Hamiltonian H_0 is just a harmonic oscillator with 'moment of inertia $I = ma^2$ and frequency $\Omega^2 = \frac{g}{a}$. Again we use Ω here for angular frequency of the H_0 harmonic oscillator, to avoid confusion with the angle variable ω.

The action-angle variables for H_0 are

$$\alpha = H_0 = \frac{\Omega}{2\pi} J, \qquad \omega = \nu t + \beta \tag{5.21}$$

where $\nu = \frac{\Omega}{2\pi}$, J is the action variable, and ω is the angle variable. This gives

$$\begin{aligned} \theta &= \sqrt{\frac{2\alpha}{I\omega^2}} \sin(\omega t + \delta) = \sqrt{\frac{J}{\pi I \Omega}} \sin\left[2\pi(\nu t + \beta)\right], \\ p &= \sqrt{2I\alpha}\cos(\omega t + \delta) = \sqrt{\frac{IJ\Omega}{\pi}} \cos\left[2\pi(\nu t + \beta)\right]. \end{aligned} \tag{5.22}$$

Since ω and β are linearly related, we are free to take (J, β) as our new canonical variables when using the transformation in Eq. (5.22).

If we use (J, β) as the new variables, with $J^{(0)} = J_0$ and $\beta^{(0)} = \beta_0$ as given constants fixed by the initial conditions, then in terms of the new variables

$$\Delta H = -\frac{mga}{24}\theta^4 = -\frac{J^2}{24\pi^2 I}\sin^4(2\pi(\nu t + \beta)). \tag{5.23}$$

Expanding by using the 0^{th} order solution gives

$$\begin{aligned} \dot{\beta}^{(1)} &= \left.\frac{\partial \Delta H}{\partial J}\right|_0 = -\frac{J_0}{12\pi^2 I}\sin^4(2\pi(\nu t + \beta_0)), \\ \dot{J}^{(1)} &= -\left.\frac{\partial \Delta H}{\partial \beta}\right|_0 = \frac{J_0^2}{3\pi I}\sin^3(2\pi(\nu t + \beta_0))\cos(2\pi(\nu t + \beta_0)). \end{aligned} \tag{5.24}$$

These results can be integrated to give $\beta^{(1)} = \beta^{(1)}(J_0, \beta_0, \nu, t)$ and $J^{(1)} = J^{(1)}(J_0, \beta_0, \nu, t)$. Before we consider computing these functions, lets pause to characterize two types of solution that occur in a more general context than simply this example.

Often we can characterize the nature of the perturbative solution without requiring a full study of the analytic form of a solution. A common situation where this is the case is when H_0 exhibits *periodic orbits* (as in the harmonic oscillator) with some frequency ν. In this case a relevant question is the following: what cumulative effect does the small perturbation have after going through one or more periods $T = \frac{1}{\nu}$? There are two possibilities:

- The perturbation itself could be *periodic*, where the parameter returns to its initial value. Here the perturbed trajectory looks much like the unperturbed one.
- Alternatively, we could have a net increment in the parameter after each orbit, called a *secular* change. After many periods, the parameter will be quite different from its value in H_0.

Example Returning to our pendulum from before, the interesting quantity to study is the average over one period of the time rate of change of the variable,

$$\overline{\dot{J}^{(1)}} = \frac{1}{T}\int_0^T \dot{J}^{(1)}(t)\,dt = \frac{J^{(1)}(T) - J^{(1)}(0)}{T}\,, \tag{5.25}$$

since this tells us how much the variable changes over one period. For our example $\overline{\dot{J}^{(1)}} = 0$ because $\int_0^{2\pi} \sin^3(\theta)\cos(\theta)\,d\theta = 0$, and therefore the perturbation to J is periodic. Actually, from integrating Eq. (5.24) we have

$$J^{(1)}(t) = J_0 + \frac{J_0^2}{24\pi^2 I\nu}\sin^4(2\pi(\nu t + \beta_0))\,. \tag{5.26}$$

Note from Eq. (5.22) that J determines the amplitude for $\theta(t)$ and $p(t)$. A comparison between the trajectory with J_0 and with $J^{(1)}(t)$ is made in Fig. 5.1, where for this figure we set $\beta_0 = 0$.

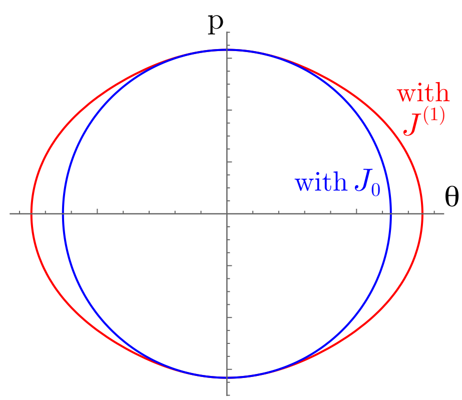

Figure 5.1: Comparison of the pendulum's periodic phase space trajectory using J_0 and $J^{(1)}(t)$.

In contrast, using $\int_0^{2\pi} \sin^4(\theta)\,\frac{d\theta}{2\pi} = \frac{3}{8}$, we find

$$\overline{\dot{\beta}^{(1)}} = \frac{1}{T}\int_0^T \dot{\beta}^{(1)}(t)\,dt = \frac{\beta^{(1)}(T) - \beta^{(1)}(0)}{T} = -\frac{J_0}{32\pi^2 I}\,, \tag{5.27}$$

which means β experiences a secular change. After many periods ($t \gg T$) the change continues to build up, and we have on average that $\beta^{(1)}(t) \approx \overline{\dot{\beta}^{(1)}}\, t + \beta_0$. (If we look at the exact solution for $\beta^{(1)}(t)$ then it has precisely this linear term in t, plus terms that are periodic over the period τ, and that is what we mean by the $\approx$ here.) Looking back at how the $\beta(t)$ dependence appears in $\theta = \theta(J, \beta, t)$ and $p = p(J, \beta, t)$ from Eq. (5.22), we see that on average the 1st order perturbation simply shifts the frequency to $\nu' = \nu + \overline{\dot{\beta}^{(1)}}$.

Recall that we determined the full frequency $\nu_{\text{full}}(E)$ numerically as an example in our study of action-angle variables, which is shown below in Figure 5.2. Recalling that $J_0 =$

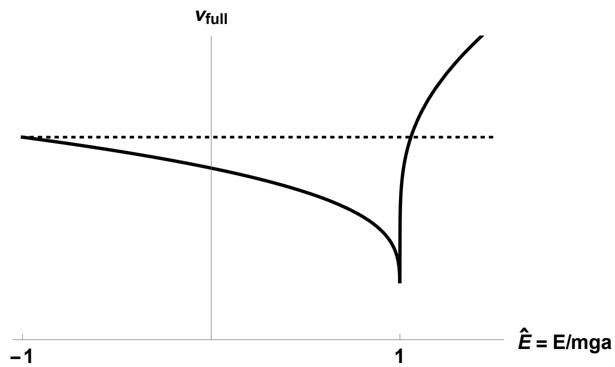

Figure 5.2: The full frequency ν_{full} vs. $\hat{E}$

$\frac{H_0}{\nu} = \frac{E+mga}{\nu}$, we can write our perturbative shift to the frequency as a function of energy

$$\nu' - \nu = \overline{\dot{\beta}^{(1)}} = -\frac{(E + mga)}{32\pi^2 m a^2 \nu}\,. \tag{5.28}$$

This is the first order correction to $\nu_{\text{full}}(\text{E})$ when it is expanded about the simple harmonic oscillator minimum at $E = -mga$, which in Fig. 5.2 gives the negative linear correction to the frequency that occurs just above $\hat{E} = E/(mga) = -1$.

5.3 Perihelion Precession from Perturbing a Kepler Orbit

Kepler Example: Consider a central force perturbation

$$H = \underbrace{\frac{1}{2m}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) - \frac{k}{r}}_{H_0}\ \underbrace{-\frac{h}{r^n}}_{\Delta H} \tag{5.29}$$

where the coupling parameter h is small and $n \geq 2$. The action-angle variables for H_0 are

$$
\begin{aligned}
J_1 &= J_\phi & \omega_1 &= \omega_\phi - \omega_\theta \\
J_2 &= J_\theta + J_\phi & \omega_2 &= \omega_\theta - \omega_r \\
J_3 &= J_r + J_\theta + J_\phi & \omega_3 &= \omega_r
\end{aligned}
$$

where only $\dot{\omega}_3 = \nu_r \neq 0$, and all the others are constant in time. The pairs (ω_i, J_i) for $i \in \{1,2,3\}$ are all canonically conjugate. One way to see this is to note that we can implement a change of variables from the canonical pairs $\{(\omega_r, J_r), (\omega_\theta, J_\theta), (\omega_\phi, J_\phi)\}$ that we considered earlier, to these variables by using the generating function

$$F_2 = (\omega_\phi - \omega_\theta)J_1 + (\omega_\theta - \omega_r)J_2 + \omega_r J_3. \tag{5.30}$$

Let us study the perihelion precession, with the parameter $\omega = 2\pi\omega_2$ determining the perihelion angle. Some examples of precession are shown in Fig. 5.3, where in the case of a planet, the precession is like that of Fig. 5.3(b) with the sun at the focus of the ellipse. We

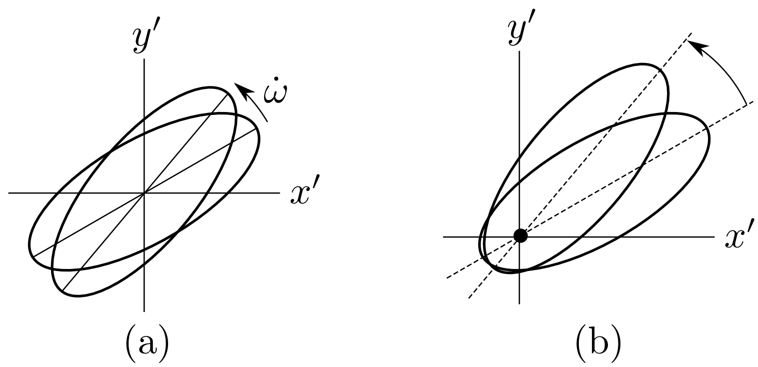

Figure 5.3: Precession of the perihelion from the point of view of coordinates centered on (a) the center of the ellipse, and (b) the focus of the ellipse. The latter is relevant for a planet like mercury orbiting the sun (shown with exaggerated eccentricity here).

know that

$$\dot{\omega}_2 = \frac{\partial \Delta H}{\partial J_2}, \qquad J_2 = J_\theta + J_\phi = 2\pi\alpha_\theta = 2\pi\ell \tag{5.31}$$

where $\ell = |\mathbf{L}|$ is the magnitude of the angular momentum. From the equation of motion $\dot{w}_2 = \partial\Delta H/\partial J_2$ we therefore have

$$\dot{\omega} = 2\pi\frac{\partial \Delta H}{\partial J_2} = \frac{\partial \Delta H}{\partial \ell}, \tag{5.32}$$

and perturbatively, $\dot{\omega}^{(1)} = \frac{\partial \Delta H}{\partial \ell}\big|_0$. We can average over the orbit to find the secular change:

$$\overline{\dot{\omega}^{(1)}} = \frac{1}{T}\int_0^T \frac{\partial \Delta H}{\partial \ell}\bigg|_0 dt. \tag{5.33}$$

Recall for the Kepler problem that $E^{(0)} = -(2\pi^2 k^2 m)/J_3^2$, and that at 0^{th} order the period is

$$T = \frac{1}{\nu_r} = \pi k \sqrt{\frac{m}{-2E^{(0)3}}} = T(J_3). \tag{5.34}$$

Therefore $\frac{\partial}{\partial \ell} = 2\pi \frac{\partial}{\partial J_2}$ does not act on $T = T(J_3)$, so we can pull the partial derivative outside the integral,

$$\overline{\dot{\omega}^{(1)}} = \frac{\partial}{\partial \ell}\left(\frac{1}{T}\int_0^T \Delta H|_0 \, dt\right) = \frac{\partial}{\partial \ell}\,\overline{\Delta H|_0}\,. \tag{5.35}$$

Thus we must calculate the average of ΔH over one period,

$$\overline{\Delta H|_0} = -\frac{h}{T}\int_0^T \frac{dt}{r^n(t)}\,. \tag{5.36}$$

Using $\ell = mr^2\dot{\psi}$ so that $dt = \frac{mr^2}{\ell}\,d\psi$ to switch variables from t to ψ, and then using the orbital equation for $r(\psi)$ we have

$$\overline{\Delta H|_0} = -\frac{hm}{\ell T}\int_0^{2\pi} \frac{d\psi}{r^{n-2}(\psi)} = -\frac{hm}{\ell T}\left(\frac{mk}{\ell^2}\right)^{n-2}\int_0^{2\pi}\big[1+\epsilon\cos(\psi-\psi')\big]^{n-2}\,d\psi\,, \tag{5.37}$$

where the eccentricity $\epsilon = \sqrt{1+\frac{2E\ell^2}{mk^2}}$ also depends on ℓ. There are two simple cases where can perform this integral:

- If $n=2$, then $\overline{\Delta H} = -\dfrac{2\pi hm}{\ell T}$, so $\overline{\dot{\omega}^{(1)}} = \dfrac{2\pi hm}{\ell^2 T}$.
- If $n=3$, then $\overline{\Delta H} = -\dfrac{hkm^2}{\ell^3 T}\int_0^{2\pi} d\psi(1+\epsilon\cos(\psi-\psi'))$, where the cos term vanishes upon integration, so $\overline{\dot{\omega}^{(1)}} = \dfrac{\partial\overline{\Delta H|_0}}{\partial \ell} = \dfrac{6\pi m^2 hk}{\ell^4 T}$.

The latter type of potential ($n=3$) is induced by corrections from general relativity to the Newtonian potential. The Schwarzschild metric is

$$ds^2 = -c^2(d\tau)^2 = \left(1-\frac{r_{\mathrm{S}}}{r}\right)c^2(dt)^2 - \frac{(dr)^2}{1-\frac{r_{\mathrm{S}}}{r}} - r^2(d\theta)^2 - r^2\sin^2(\theta)(d\phi)^2 \tag{5.38}$$

for $r_s = \frac{2GM}{c^2}$ where M is the central mass (say of the sun), G is Newton's gravitational constant, and c is the speed of light. The geodesic equation for radial motion is given by

$$E = \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \tag{5.39}$$

where $V_{\rm eff}(r) = -\frac{mc^2 r_S}{2r} + \frac{\ell^2}{2mr^2} - \frac{r_S \ell^2}{2mr^3}$. (More background details on the Schwarzchild metric and the derivation of this geodesic equation are discussed below in the Side Note on page 102.) Defining $k = GMm$, then the effective potential can be rewritten as

$$V_{\rm eff}(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} - \frac{k\ell^2}{c^2m^2r^3}, \tag{5.40}$$

from which we can identify $h = \frac{k\ell^2}{c^2m^2}$. Note that h must be treated as a constant independent of the canonical variable ℓ for the purpose of the above perturbative analysis (we simply substitute this value for h at the end).

For Mercury, $T = 0.2409 T_{\rm Earth}$, $\epsilon = 0.2056$, and $a = 5.79 \times 10^7\,$km, while $\frac{GM_{\rm sun}}{c^2} = 1.4766\,$km, so we get a precession rate of $\overline{\dot{\omega}^{(1)}} = 42.98\,$arcseconds/century from general relativity. (An arcsecond is 1/3600'th of a second.) After removing other contributions, such as a shift of 531.54 arcseconds/century from perturbations by other planets, the data on mercury's orbit shows a shift of 43.1 arcseconds/century (excellent agreement!). This was historically one of the first tests of general relativity, and still remains an important one.

We could also consider perturbations involving momentum variables rather than coordinates.

Example Consider the relativistic correction to harmonic oscillator where the relativistic energy

$$E = \sqrt{c^4m^2 + c^2p^2} = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots \tag{5.41}$$

In this case to analyze the first order perturbative correction we take

$$H = \underbrace{\frac{1}{2m}\left(p^2 + m^2\Omega^2 q^2\right)}_{H_0} \underbrace{-\frac{p^4}{8c^2m^3}}_{\Delta H}. \tag{5.42}$$

From H_0, the variables have a canonical transformation from the H-J analysis that gives

$$q = \sqrt{\frac{J}{\pi m\Omega}}\sin(2\pi(\nu t + \beta))\,, \qquad p = \sqrt{\frac{Jm\Omega}{\pi}}\cos(2\pi(\nu t + \beta))\,. \tag{5.43}$$

This gives

$$\Delta H = -\frac{J^2\Omega^2}{8\pi^2c^2m}\cos^4\left[2\pi(\nu t + \beta)\right]. \tag{5.44}$$

Since $\dot{J}^{(1)}$ is odd over one period, it turns out that J is periodic once again

$$\dot{J}^{(1)} = -\frac{\partial \Delta H}{\partial \beta}\bigg|_0 \quad \Rightarrow \quad \overline{\dot{J}^{(1)}} = 0. \tag{5.45}$$

Meanwhile, the change for $\dot{\beta}^{(1)}$ is secular,

$$\dot{\beta}^{(1)} = \frac{\partial \Delta H}{\partial J}\bigg|_0 = -\frac{J_0\Omega^2}{4\pi^2mc^2}\cos^4\left[2\pi(\nu t + \beta)\right] \quad \Rightarrow \quad \overline{\dot{\beta}^{(1)}} = -\frac{3J_0\Omega^2}{32\pi^2mc^2}\,. \tag{5.46}$$

Thus, $\overline{\dot{\beta}^{(1)}}$ from the relativistic correction ΔH is again a negative shift to the frequency of the oscillator.

This ends our discussion of perturbation theory in classical mechanics.

Side Note: The Schwarzchild Geodesic from Action Minimization

The Schwarzchild metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{r_s}{r}\right) c^2 t^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \,, \tag{5.47}$$

where $r_s = \frac{2GM}{c^2}$ is the Schwartzchild radius. The geodesic orbit for a test particle is a curve which minimizes proper distance with this metric. In this case, we have

$$0 = \delta s = \delta \int ds = \delta \int \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{\frac{1}{2}} d\tau \tag{5.48}$$

where τ is the proper time and $ds^2 = c^2 d\tau^2$. (One method of determining the geodesic equations is to use $0 = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$ with the Christoffel symbols $\Gamma^\lambda_{\mu\nu}$ determined from the metric, but we will follow a different approach.)

The minimization in Eq. (5.48) is equivalent to applying the minimal action principal for the Lagrangian

$$\begin{aligned} L &= \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \\ &= \frac{m}{2} \left[\left(1 - \frac{r_s}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{\left(1 - \frac{r_s}{r}\right)} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 - r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2 \right] \end{aligned} \tag{5.49}$$

since the presence of the extra square root $(\ldots)^{\frac{1}{2}}$ does not matter for this minimization. Here we have the generalized coordinates $x^\mu = (t, r, \theta, \phi)$ which are to be considered as functions of the proper time variable τ. Also, the mass m is a test mass (which also gives us the proper units).

Because t and ϕ are cyclic variables in L, we have

$$\begin{aligned} p_t &= \frac{\partial L}{\partial \dot{t}} = m \left(1 - \frac{r_s}{r}\right) c^2 \frac{dt}{d\tau} = E^{\rm tot} && \text{energy} \\ p_\phi &= -\frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2\theta \frac{d\phi}{d\tau} = \ell && \text{angular momentum} \end{aligned} \tag{5.50}$$

Using the E-L equation on θ we obtain

$$
\begin{aligned}
0 &= \frac{d}{d\tau}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} \\
&= \frac{d}{d\tau}\left(mr^2\frac{d\theta}{d\tau}\right) - mr^2 \sin\theta\cos\theta\dot{\phi}^2
\end{aligned}
\tag{5.51}
$$

where planar motion with $\theta = \frac{\pi}{2}$ is a solution that suffices for our purposes. Now

$$
\frac{ds^2}{d\tau^2} = c^2 = \left(1 - \frac{r_s}{r}\right)c^2\frac{E_{\text{tot}}^2}{m^2\left(1-\frac{r_s}{r}\right)^2 c^4} - \frac{1}{\left(1-\frac{r_s}{r}\right)}\left(\frac{dr}{d\tau}\right)^2 - r^2\frac{\ell^2}{m^2 r^4}
\tag{5.52}
$$

gives a radial equation. Separating out the rest mass and expanding for $E \ll mc^2$ we have:

$$
E^{\text{tot}} = mc^2 + E \quad \Rightarrow \quad \left(E^{\text{tot}}\right)^2 \approx m^2c^4 + 2mc^2E
\tag{5.53}
$$

Finally,

$$
\underbrace{\frac{m}{2}\left(c^2 + \frac{l^2}{m^2r^2}\right)\left(1 - \frac{r_s}{r}\right) - \frac{m}{2}c^2}_{V_{\text{eff}}(r)} + \frac{m}{2}\left(\frac{dr}{d\tau}\right)^2 = E
\tag{5.54}
$$

Note that the rest mass $mc^2/2$ terms cancel. Therefore the effective potential is

$$
\begin{aligned}
V_{\text{eff}}(r) &= -\frac{r_s mc^2}{2r} + \frac{l^2}{2mr^2} - \frac{r_s l^2}{2mr^3} \\
&= -\frac{k}{r} + \frac{l^2}{2mr^2} - \frac{kl^2}{m^2c^2r^3}
\end{aligned}
$$

where $r_s = \frac{2GM}{c^2}$ and $k = GMm$, so $mr_sc^2 = 2k$. This is the result that was quoted above in Eq. (5.40).