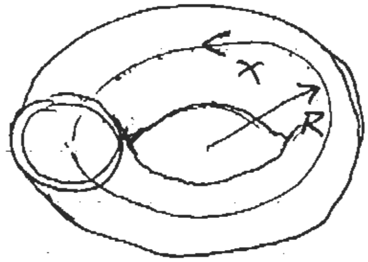
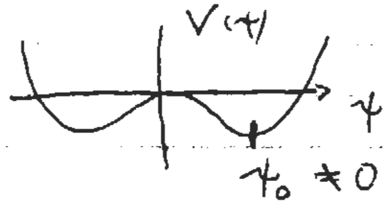
✱ Superfluidity



$$\Omega = \int d^3x \; \frac{\hbar^2}{2m} |\nabla \psi|^2 - \mu |\psi|^2 + \frac{1}{2} v_0 |\psi|^4$$



density current

$$J_x = v_x \rho = \frac{1}{m} \mathrm{Re}\left(\psi^* \frac{\hbar}{i} \partial_x \psi\right)$$

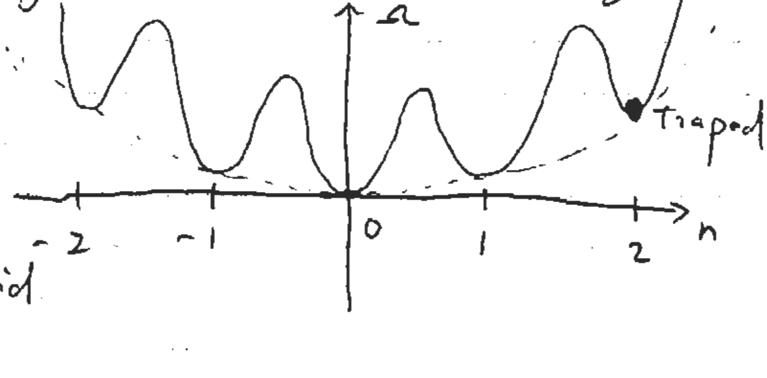
if $\psi = \psi_0 = \text{const.}$ $\quad J_x = 0$ $\quad$ no flow

if $\psi = \psi_0 e^{i \frac{n}{R} x}$ (← integer) $= \frac{n}{\hbar R m} |\psi_0|^2$ $\quad$ flow $\neq 0$

Key: $\psi_0 e^{i\frac{n}{R}x}$ minimizes Ω and satisfies the eq. of motion.

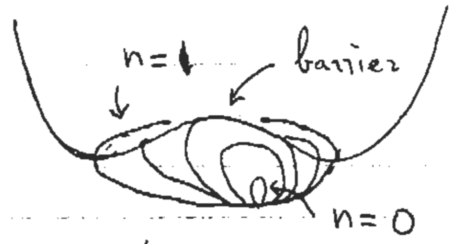
In order for the flow to decay to zero n must decay to zero. But n is quantized as integer and cannot decay

Thus the fluid keep flowing $\Rightarrow$ superfluid



In order for $n=1 \Rightarrow n=0$

or $\psi = e^{i\frac{x}{R}}\psi_0 \Rightarrow \psi_0$

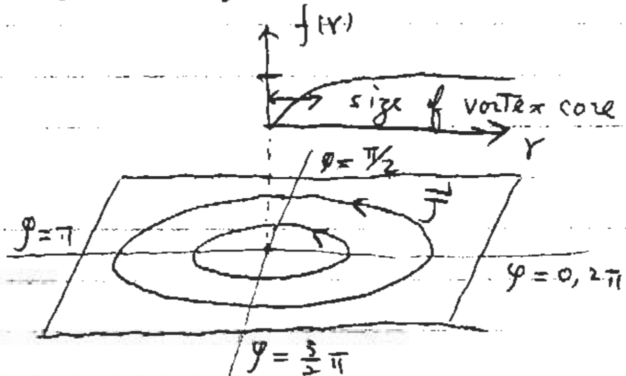


ψ must pass $\psi=0$

potential barrier, n cannot change

Only vortex tunneling can change n and reduce the superfluid flow.

★ What is votex



$$\psi = f(r)\,\psi_0\, e^{i\varphi}$$

$(x = r\cos\varphi,\ y = r\sin\varphi)$

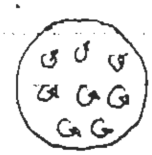
velocity $\vec{v}_s = \frac{\hbar}{m}\nabla\varphi(x,y)$

density current $\vec{J} = n\,\vec{v}_s$

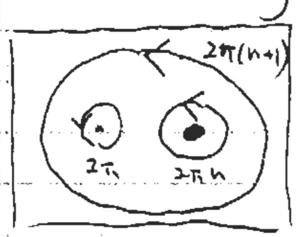
quantization of vorticity

$$\oint d\vec{s}\cdot\vec{v}_s = \frac{h}{2m}\cdot 2\pi\cdot \text{int} = \frac{h}{m}\times\text{integer}$$

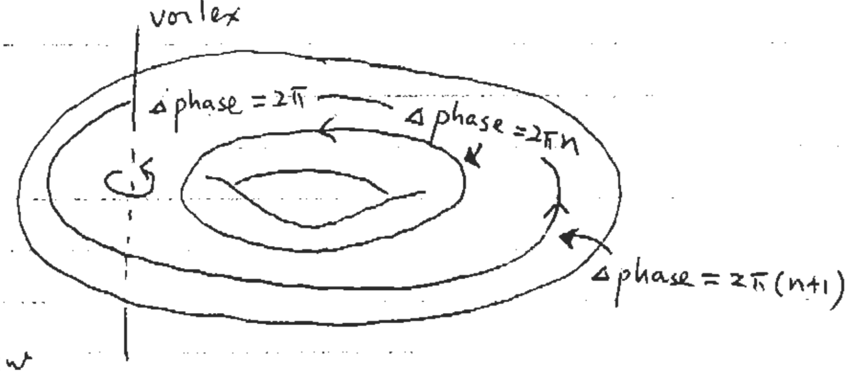
No uniform rotation: $\vec{v}(\vec{r}) = \vec{\omega}\times\vec{r}$ in superfluid!



Tunneling:



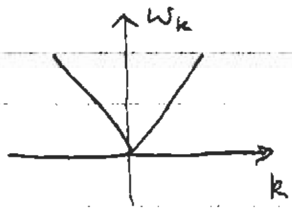
top view



★ Excitation on a flowing superfluid

$\psi = \psi_0$, $v_s = 0$

$\omega_k = v|k|$

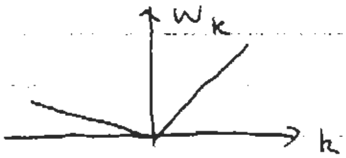


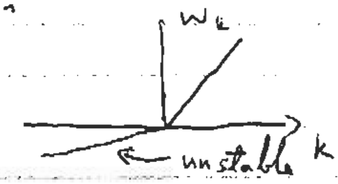
$\psi = \psi_0 e^{iKx}$, $v_s = \frac{\hbar K}{m}$

$$\omega_k = \begin{cases} (v + v_s)|k| & k > 0 \\ (v - v_s)|k| & k < 0 \end{cases}$$

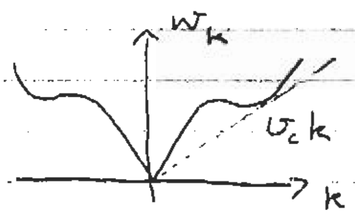
$$= v|k| + v_s k$$

stable flow no friction

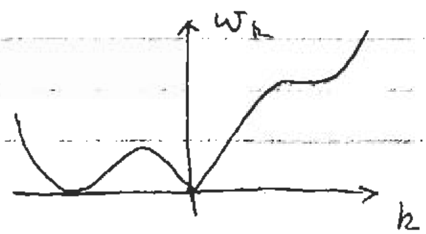




Real system



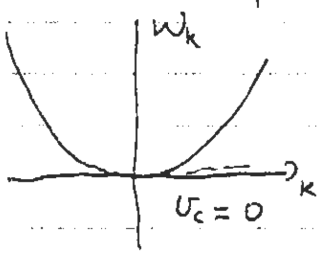
$v_s = 0$



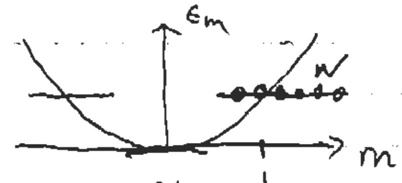
$v_s = v_c$

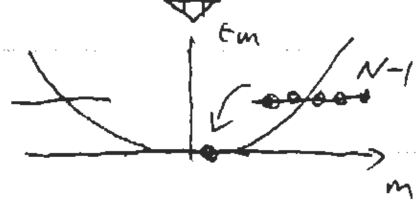
v_c critical velocity of superfluid flow.

★ No superfluid flow for free boson condensed state



$\psi = e^{im\frac{x}{R}}$





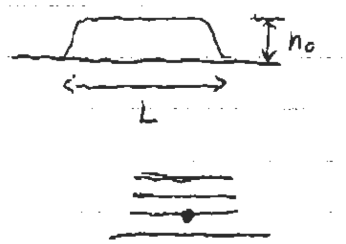
* Remark

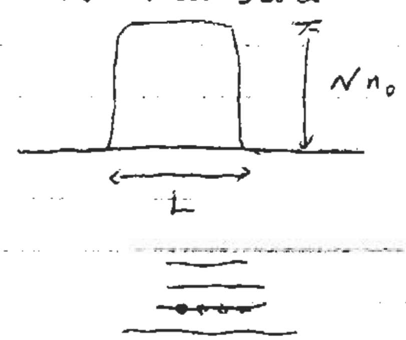
Boson condensation : single-particle state $\psi_0(x)$

Boson condensed state = all bosons are in ψ_0 state

$\Rightarrow$ N- boson wave function $\Psi(x_1, \cdots x_N) = \psi_0(x_1) \cdots \psi_0(x_N)$

density : single-particle state , N- boson state





Collective excitations :

change $\psi_0(x) \rightarrow \psi_1(x)$ (ψ_1 & ψ_0 do not have to be normal to each other)

New N- boson state

$$\Psi_1(x_1 \cdots x_N) = \psi_1(x_1) \cdots \psi_1(x_N)$$

Two types of collective modes: (ground state $\psi_0 = \text{const.}$)

a) density wave $\psi_1 = \psi_0 + \alpha e^{ik\cdot x}$ 

b) vortex (minimum) $\psi_1 = f(r)\, e^{i\theta}\, \psi_0$

(r, θ, z) polar coordinate

n- vortex $\psi_1 = f(r)\, e^{in\theta}\, \psi_0$

★ G-L theory of boson condensation

Boson condensed state: All bosons are in the same single-particle state $\psi(x)$.

(ie N-Boson wave function $\Psi(x_1 \cdots x_N) = \psi(x_1) \cdots \psi(x_N)$)

order parameter = amplitude of condensed bosons.

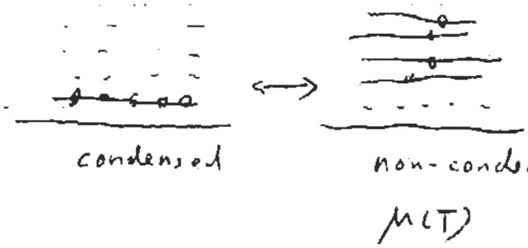
At $T=0$ the free energy (or energy) of interacting boson

$$A = \int d^3x \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + (U-\mu)|\psi|^2 + \frac{U_0}{2}|\psi|^4 \right]$$

For finite T. ψ amplitude of condensed bosons

$$A = \int d^3x \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + a(T)|\psi|^2 + \frac{U_0}{2}|\psi|^4 \right] + A_0(T)$$

picture:

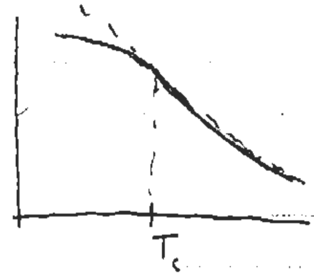


$a(T) > 0$ $\quad \psi = 0$ no condensation

$a(T) < 0$ $\quad \psi \neq 0$ finite condensation

A $U(1)$ symmetry $\psi \to e^{i\theta}\psi$ $\quad A \to A$

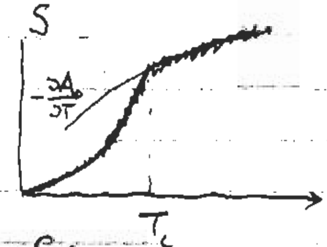
Minimize $\quad a(T)\,|\psi|^2 + \frac{v_0}{2}|\psi|^4$



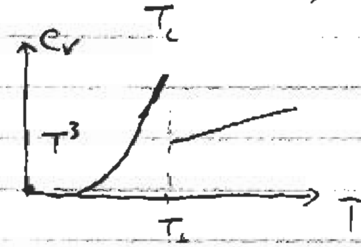
$$\Rightarrow \quad a(T) + v_0|\psi|^2 = 0$$

$$\Rightarrow \quad \frac{A}{V} = \frac{A_0}{V} + \begin{cases} 0 & a(T) > 0 \\ -\frac{1}{2}\frac{a^2}{v_0} & a(T) < 0 \end{cases}$$

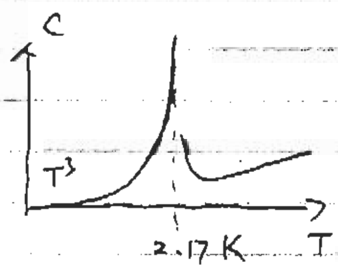
$$S = -\frac{\partial A}{\partial T} \; :$$



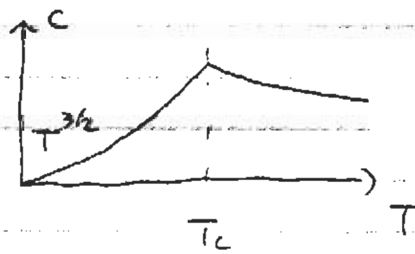
$$C = T\frac{\partial S}{\partial T} \; :$$



real He^4



free bosons



✯ Solid as "boson condensation" (CDW)

liquid   solid

what is order parameter?

$$n(x) = n_0 + \alpha \cos(kx+\phi) = n_0 + Re\,(\psi e^{ikx})$$

$\uparrow$ complex number.

ψ is order parameter for a solid (charge-density-wave) (CDW)

$\psi = 0 \Rightarrow$ no CDW

phase of $\psi \Rightarrow$ position of CDW

Translation symmetry $\Rightarrow$ energy does not depend on the position of CDW

$\Rightarrow$ free energy does not depend on the phase of ψ (U(1) symmetry)

$$\Rightarrow \text{G-L theory} \quad A = \int d^D x \;\; a(T)\,|\psi|^2 + b(T)\,|\psi|^4 + \dots$$

no $\psi\psi^*$, $\psi\psi$ + $\psi^*\psi^*$ terms since they break the U(1) symmetry

CDW = Boson condensation

$\psi(\vec{x}) = f(r)\,e^{i\theta}\,\psi_0 \Rightarrow$ vortex in BC

what $n(x) = n_0 + Re\,(\psi(\vec{x})\, e^{i\vec{k}\cdot\vec{x}})$ looks like?