V. Random variable

① One random variable ... s

Prob. distribution $P(s)$, $\sum_s P(s) = 1$

Average $\langle s \rangle = \sum_s s\, P(s)$

moments $\langle s^n \rangle = \sum_s s^n P(s)$

Fluctuation:

$$\Delta s = \sqrt{\langle (s - \langle s \rangle)^2 \rangle} = \sqrt{\langle s^2 \rangle - \langle s \rangle^2}$$

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② Two random variables $s_1, s_2 = 0, 1$

$P(s_1 = 1) = P_1(1)$ $\qquad P(s_1 = 0) = 1 - P_1(1) = P_1(0)$

$P(s_2 = 1) = P_2(1)$ $\qquad P(s_2 = 0) = 1 - P_2(1) = P_2(0)$

Joint prob.

$$P(s_1 = 1 \,\&\, s_2 = 1) = \cancel{P_1(1) P_2(1) ?}$$

$$= P_{12}(1,1)$$

$\boxed{P_{12}(s_1, s_2)}$ Joint prob.

$$P_{12}(11) + P_{12}(10) + P_{12}(01) + P_{12}(00) = 1$$

$$\underbrace{P_1(1)}_{s_1 = 1} = \underbrace{P_{12}(1,1) + P_{12}(1,0)}_{\text{regardless } s_2}$$

Conditional randomness.

$$P_{\text{if } s_2=1}(s_1) \propto P_{12}(s_1, 1)$$

$$P_{\text{if } s_2=1}(s_1) = \frac{P_{12}(s_1, 1)}{\sum_{s_1=0,1} P_{12}(s_1, 1)}$$

$$P_{\text{if } s_2=0}(s_1) = \frac{P_{12}(s_1, 0)}{\sum_{s_1=0,1} P_{12}(s_1, 0)}$$

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example. I roll a dice $n = 1, \ldots, 6$

$s_1 = 1$ if n is multiple of 2 $\quad s_1 = 0$ otherwise

$s_2 = 1$ if n is multiple of 3 $\quad s_2 = 0$ otherwise

$P_1(1) = \frac{1}{2} \qquad P_1(0) = \frac{1}{2}$

$P_2(1) = \frac{1}{3} \qquad P_2(0) = \frac{2}{3}$

$P_{12}(11) = \frac{1}{6}\Big|_{n=6} \qquad P_{12}(10) = \frac{1}{3}\Big|_{n=2,4}$

$P_{12}(01) = \frac{1}{6}\Big|_{n=3} \qquad P(00) = \frac{1}{3}\Big|_{n=1,5}$

$P_{\text{if } s_2=1}(s_1) = \frac{1/6}{1/6+1/6}, \frac{1/6}{1/6+1/6} = \frac{1}{2}, \frac{1}{2}$

equal.

$P_{\text{if } s_2=0}(s_1) = \frac{1}{2}, \frac{1}{2}$

$P_{\text{if } s_1=1}(s_2) = \underset{s_2=1}{\frac{1}{3}}, \underset{s_2=0}{\frac{2}{3}}$

equal

$P_{\text{if } s_1=0}(s_2) = \frac{1}{3}, \frac{2}{3}$

$\boxed{s_1 \,\&\, s_2 \text{ are independent random variables}}$

Example II.

$s_1 = 1$ if $n = 1,2,3$ $\quad s_1 = 0$ if $n = 4,5,6$

$s_2 = 1$ if $n \% 2 = 0$ $\quad s_2 = 1$ if $n\% 2 \neq 0$

$$P_1(s_1) = \frac{1}{2}, \frac{1}{2}$$
$$P_2(s_2) = \frac{1}{2}, \frac{1}{2}$$

	$s_2 = 1$	$s_2 = 0$	
$P_{12}(s_1, s_2) =$	$\frac{1}{6}$	$\frac{1}{3}$	$s_1 = 1$
	$\frac{1}{3}$	$\frac{1}{6}$	$s_1 = 0$

$$P_{\text{if } s_2 = 1}(s_1) = \frac{1}{3}, \frac{2}{3}$$
$$P_{\text{if } s_2 = 0}(s_1) = \frac{2}{3}, \frac{1}{3}$$

not equal.

s_1 & s_2 are not independent

If s_1 & s_2 are independent

we can obtain $P_{12}(s_1, s_2)$ from $P_1(s_1)$ & $P_2(s_2)$

$$\therefore \boxed{P_{12}(s_1, s_2) = P_1(s_1) P_2(s_2)}$$

$$\langle s_1 s_2 \rangle = \langle s_1 \rangle \langle s_2 \rangle$$

Random force

$\langle F(s) F(0) \rangle =$
$= \langle F(s) \rangle \langle F(0) \rangle = 0$
if $s \to \infty$

$$\langle s_1 s_2 \rangle = \sum_{s_1, s_2} s_1 s_2 P_{12}(s_1, s_2)$$
$$= \sum_{s_1, s_2} s_1 P_1(s_1)\, s_2 P_2(s_2)$$
$$= \left(\sum_{s_1} s_1 P_1(s_1)\right)\left(\sum_{s_2} s_2 P_2(s_2)\right) = \langle s_1 \rangle \langle s_2 \rangle$$

for correlated random variable $\langle s_1 s_2 \rangle \neq \langle s_1 \rangle \langle s_2 \rangle$

$\langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle \equiv$ correlation between s_1 & s_2.

③

Binomial distribution

Consider n independent random variables

$$s_1, \cdots, s_n \quad , \quad s_i = 0, 1$$

$$P(s_i = 1) = p \quad , \quad P(s_i = 0) = 1 - p$$

Prob. the set to take a particular values

$$P(s_1, s_2 \cdots s_n) = P(s_1) P(s_2) \cdots P(s_n)$$

Prob. for $\Sigma s_i = k$

$$B(k; n, p) = \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\begin{cases} \langle k \rangle = \sum_k k\, B(k; n, p) = p n \\ \langle k^2 \rangle = n^2 p^2 + n p (1-p) \end{cases}$$

How to calculate $\langle k \rangle$ let $x = \frac{p}{1-p}$

& $Z = (1+x)^n$

$$= 1 + n x + \frac{n(n-1)}{2} x^2 \cdots + \binom{n}{k} x^k \cdots$$

$$x \cdot \frac{d}{dx} \ln Z = \frac{1}{Z} \Sigma\, k \binom{n}{k} x^k$$

$$\frac{1}{Z} \binom{n}{k} x^k = \binom{n}{k} \frac{(p/1-p)^k}{(1+\frac{p}{1-p})^n} = \binom{n}{k} \frac{p^k (1-p)^{n-k}}{(1-p+p)^n}$$

$$= B(k; n, p)$$

$$\langle k \rangle = x \frac{d}{dx} \ln Z$$

$$= x\, n \frac{1}{1+x} = np \checkmark$$

④ Central limit theorem

Given n independent random variables

$x_1, \cdots, x_n$ with distribution $P(x)$

Let $X = \frac{1}{n} \sum_i x_i$.

In $n \to \infty$ limit, the distribution of X

is given be Gaussian distribution

$$P_G(X) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(X-\langle X\rangle)^2}{2\sigma}}$$

$$\int dX \, P_G(X) = 1$$

$$\int dX \, X \, P_G(X) = \langle X \rangle = \int dx \, x P(x) = \langle x \rangle$$

$$\int dX \, (X - \langle X \rangle)^2 \, P_G(X) = \sigma$$

$$= \frac{1}{n^2} \left\langle \left(\sum_i (x_i - \langle x \rangle) \right)^2 \right\rangle$$

$$= \frac{1}{n^2} \sum_i \langle (x_i - \langle x \rangle)^2 \rangle$$

$$+ \frac{1}{n^2} \sum_{i \neq j} \underbrace{\langle (x_i - \langle x \rangle)(x_j - \langle x \rangle) \rangle}_{\fallingdotseq 0}$$

$$\boxed{\sigma = \frac{1}{n} \langle (x - \langle x \rangle)^2 \rangle}$$