**PROFESSOR:** So we have this integral. And with-- let me go here, actually. With the counter gamma equal to C1, this counter over here, and the constant c equal to 1. So C1 and the constant c equal to 1. This psi that we have defined, psi of u, is in fact the airy function of u. A i of u. I is not-- I think I tend to make that mistake. I doesn't go like a subscript. It's A i like the first two letters of the name. So that's the function A i of u.

And now you could ask, well, what is the other solution? Now, in fact, this diagram suggests to you that there's other solutions because you could take other counters and make other solutions. In fact, yes, there are other ways. For example, if you did a counter-- do I have a color? Other color? Yes. If you did a counter like this, yellow and yellow, this is not the same solution. It is a solution because of the general argument and because the endpoints are in these regions where things vanish at infinity.

So the yellow thing is another solution of the differential equation. So the other airy function is defined, actually, with this other counter. It's defined by taking the yellow counter like this. This is going to be called C2. A counter like that. It just comes parallel to this one and then goes down.

And, actually, in order to have a nicely defined function, one chooses for the function B i of u the following. Minus i times the integral over the counter C1 of the same integrant. So I will not copy it. Always the same integrant. Plus 2 i terms the integral over the counter C2 of the same thing.

So the B i function is a little unusual in that it has kind of a little bit of the A i function because you also integrate over C1. But you integrate as well over C2. That guarantees that-- actually, this second airy function behaves similar to A i for negative u, and while A i goes to 0 for positive u, this one will diverge.

There are expressions for this function. I'll give you an integral. 2 integrals that are famous are A i of u equals 1 over pi. Integral from 0 to infinity. d k cosine. k cubed over 3 plus k u. And for B i of u. 1 over pi-- this is a little longer. Integral from 0 to infinity as well. d k. And you have an exponential of minus k cubed over 3 plus k u, plus the sine of k cubed over 3, plus k u. And that's it. It's kind of funny. One is the cosine and one is the sine. And it has this extra different factors.

So these are your two functions. And of the relevance to our w k b problem is that they're necessary to connect the solutions, as we will see. But we need a little more about these functions. We need to know there are asymptotic behaviors. Now there are functions, like the exponential function, that has a Taylor series, e to the z. 1 plus z plus z squared over 2. And it's valid, whatever these angles-- the argument of z is. That's always the same asymptotic expansion, or the same Taylor series.

For this functions, like the airy function, for some arguments of u, there's one form to the asymptotic expansion. And for some other arguments, there's another form. That's sometimes called the Stokes phenomenon. And for example, the expansion for positive u is going to be a decaying exponential here. But for negative u, it will be oscillatory.

So it's not like a simple function, like the exponential function has a nice, simple expansion everywhere. It just varies. So one needs to calculate this asymptotic expansions, and I'll make a small comment about it of how you go about it. The nice thing about these formulas is that they allow you to understand things intuitively and derive things yourself.

Here you see the two airy functions. They make sense. The other thing that is possible to do with this counter representations is to find the asymptotic expansions of these functions. And they don't require major mathematical work. It's kind of nice.

So let's think a little about one example. If you have the airy function A i of u that is of the form integral 1 over-- well, it begins v k over 2 pi. The integral over counter C1. Maybe I should have done them curly. Curly C1 would have been clearer. e to the i. k cubed over 3 plus k u. That is your integral.

And this is the phase of integration. Phase. Now, in order to find the asymptotic expansion for this thing, we'll use a stationary phase condition. So the integral is dominated by those places where the phase is stationary. So the 5 prime of k is k squared plus u. And we want this to be equal to 0.

So take, for example, u positive. Suppose you want to find the behavior of the airy function on the right of the axis. Well, this says k squared is equal to u. So k squared is equal to minus u. And that's-- the right hand side is negative because u is positive. And therefore, k-- the points where this is solved are k equal plus minus i square root of u. So where are those points in the k plane? They're here and there. And those are the places where you get stationary phase.

So what can you do? You're supposed to do the integral over this red line. C1. Well, as we argued, this can be lifted and we can make the integral pass through here. You can now do this integral over here. It's the same integral that you had before.

In this line, we can say that k is equal to i square root of u plus some extra k tilde. We say here is i square root of u for u positive. There is this other place, but that-- we cannot bring the counter down here because in this region, the end points still contribute. So we cannot shift the counter down, but we can shift it up.

So we have to do this integral. So what happens? You can evaluate that phase 5k under these conditions. It is a stationary phase point-- this one-- so the answer is going to have a part independent of k tilde, a linear part, with respect to k tilde that will vanish because at this point the phase is stationary. And then a quadratic part with respect to k tilde.

So the phase, 5k, which is k cubed over 3 plus k u. When you substitute this k here, it's going to give you an answer. And the answer is going to be 2 over 3 i u to the 3/2, plus i square root of u k squared tilde, plus k tilde cubed over 3. That's what you get from the phase.

You say, well, that's pretty nice because now our integral psi has become the integral d k tilde over 2 pi. So you pass from k to k tilde variable. e to the minus 2/3. u to the 3/2. That is because you have i times 5k, so you must multiply by i here. Then minus square root of u k squared. Then plus k cubed. OK. Tildes of course.

**AUDIENCE:** [INAUDIBLE]

**PROFESSOR:** Um. Yes. i k. This should be like that. Yes. OK. So now what happens? If u is large enough, this quantity over here is going to mean that this integral is highly suppressed over k tilde. It's a Gaussian with a very narrow peak. And therefore, we can ignore this term. This term is a constant. So what do we get from here? We get 1 over 2 pi from the beginning. This exponential. e to the minus 2 over 3. u to the 3/2. 3/2. And then from this Gaussian, we get square root of pi over square root of square root of u. So it's u to the 1/4.

So I think I got it all correct. And therefore, the function A i of u goes like-- or it's proportional to this decaying exponential. Very fast decaying exponential. 1 over 2. 1 over square root of pi. 1 over u to the 1/4. e to the minus 2/3. u to the 3/2. And this is when u is greater than 0 and, in fact, u much greater than 1, positive and large.

So this shows you the power of this method. This is a very powerful result. And this is what we're going to need, pretty much, now in order to do our asymptotic solutions and the matching of w k b. There is an extra term, or an extra case, you can consider. What happens to this integral when u is negative? When u is negative, there's two real roots, and the stationary points occur on the real line. The integral is, therefore, a little more straightforward. You don't have to even move it.

And we have another expansion, therefore, which is A i of u is actually equal to 1 over the square root of pi. 1 over u, length of u to the 1/4. Cosine of 2/3. Length of u to the 3/2 minus pi over 4. This is for u less than 0. So the airy function becomes an oscillatory function on the left. So how does this airy function really look like?

Well, we have the true airy function behaves like this. It's a decaying exponential for u positive. And then a decaying oscillatory function for u negative. For u negative, it decays eventually. The frequency becomes faster and faster. And that's your airy function A i of u. Here is u.

Your asymptotic expansions, the ones that we found up there for u greater than 1, match this very nicely. But eventually they blow up, so they're a little wrong. And they actually match here quite OK. But here they go and also blow up because of this factor. So they both blow up, but they're both quite wrong in this area, which you would expect them to be wrong. These are the regions where this two asymptotic expansions make no sense. They were calculated on the hypothesis that u is much bigger than 1, or much less than 1. Minus 1. And therefore, you get everything under control except this area. So now let's do the real work of w k b. That was our goal from the beginning, so that's what we'll try to do now.