MITOCW | L7.4 Connection formula stated and example

PROFESSOR: So let's do a connection formula and use it to solve a problem. The derivation of such connection formulas, we'll face it the next time. And we'll go further applications of the methods. So, yes.

AUDIENCE: So [INAUDIBLE] that problem like [INAUDIBLE] where the wave function vanished. Is that a problem of the [? our ?] perturbation or a classical problem? Because at that point, then it means the kinetics and the potential are equals to each other. So like the potential cannot be bigger than the energy, then that's a classical thing. That's not a perturbation problem.

PROFESSOR: Right. This is not a problem of the perturbation theory. It's just our lack of knowledge, our ignorance of how the solution looks near there. So there's nice WKB formulas for this solution away from the turning points. Those are this. But the solutions near the turning points violate the semiclassical approximation. Therefore, you have to find the solution near the turning points by any method you have. That is not going to be the semiclassical method. And then you will find the continuation.

Now, there's several ways people do this. The most down to earth method, which is I think the method we're going to use next time, is trying to solve the thing near there, the solution. People that are more sophisticated use complex variables, methods in which they think of the solution in the complex plane, the x plane becomes complex. And they're coming to the turning point and they go off the imaginary axis to avoid it and come on the other side.

It's very elegant, very nice, harder to make very precise, and a little difficult to explain. I don't know if I'll try to do that. But it's a nice thing. It's sort of avoiding the turning point by going off the axis. It sounds crazy. So here is a connection formula. Here is x equals a. Here is v of x. And here is a solution for x less than a, little a, I'll write it like that, p of x cosine x to a kappa k of x vx prime minus pi over 4 minus B over square root of p of K of x sine x to a k of x prime vx prime minus pi over 4.

So look, this is a general solution in allowed region. [INAUDIBLE] no, it doesn't look like what you wrote here. But sines and cosines are linear combinations of these things. And why do I put this silly minus pi over 4? Because that's what my solutions connect to solutions on the other side. Which is to say that the solution on the other side, people that work this connection formulas, discovered that takes this form, a to x kappa of x prime vx prime plus b over the square root of kappa of x e exponential a to x kappa of x prime vx prime.

So here it is. Those numbers, a and b, are things you have to keep track now. They say if your solution for x greater [INAUDIBLE] has this decaying exponential and this growing exponential with b an a, your solution far to the left will look like this. It's a pretty strange statement. Sometimes you don't want b to be 0. Sometimes you do. Let's do an example with this stuff so that you can appreciate what goes on.

So that's your first look at a connection condition. There's a little bit of subtleties on how to use them. We will discuss those subtleties next time, as well. But let's use it in a case where we can make sense of this without too much trouble. So here is the example.

And somebody wants you to solve the following problem. You have some slowly varying potential that grows, grows indefinitely. v of x. And you wish to find the energy eigenstates. In particular, you wish to find the energies. What are the possible energies of this potential? This potential with have a infinite wall here. So the potential is infinite here. And it grows like that up on this side.

We're going to try to write the solution for this. So we'll do this with the WKB solution. And let me do it in an efficient way. So what do we have here? This is the analog of our point x equals a. Is that right? This is the point where the turning solution goes funny. So we will call it, for the same reason we did before, this point a will make matters clearer.

We truly don't know what that point is because we truly don't know what are the allowed energies. But so far we can write E as a variable that we don't know. And a can be determined if you know E because you know the shape of the potential.

OK. So let's think of the region, x much greater than a. We are here. We're in a forbidden region. And the potential is still slowly varying. Let's assume that slope is small. So a solution of this type would make sense.

So we must have a WKB solution on this right hand side to the right of x equals a. And that's the most general solution we'll have. On the other hand, here there's two types of solutions, a growing exponential and a decaying exponential. And we must have only the decaying exponential. Because this potential never turns.

If this potential would grow here and then turn down, you might have what is called the tunneling problem. And this has to be rethought. But this problem is still a little easier. We have just this potential growing forever. And on the right, we must have b equal to 0. For x greater than a solution, b is equal to 0. And this solution must have a different from 0.

But if the solution has a different from 0, we now know the solution on this region, x significantly less than a. In this region we know the solution is given by that formula up there. So our solution for x much less than a, the solution must take the form whatever a is. 1, 2, 3. I kind of normalized these things yet.

So the solution, I'll write it, psi of x, will be of the form 1 over square root of k of x cosine of x to a k of x prime vx prime minus pi over 4. All right. That's what WKB predicts because of the connection condition. On the forbidden region, you know which wave exists. And therefore, far to the left of the turning point, you'd also know the wave function. It's the term within a.

So have we solved the problem? Well, I don't think we have. We still don't know the energy, so we must have not solved it. In fact, it doesn't look like we've solved it at all. Because at x equals 0, these wave functions should vanish because there's a hard wall. And I don't see any reason why it would vanish.

So let's do a little work here expanding this and orienting it a little better. So I want to write this integral from 0, x to a. You're having an integral. You have 0, x, and a. Because we are in the x less than a region, this integral is the one that we have here. I'll write it as an integral from 0 to a minus an integral from 0 to x. So integral from x to a is equal to integral from 0 to a minus an integral from 0 to x. This will make things a little clearer.

In fact, we could do things still a little easier. So what do we have here? We have this exponential, the wave function. Not an exponential, a trigonometric function. 1 over square root of k of x cosine of minus the integral from 0 to x of k of x dx prime. This integral gives rise to two integrals and I wrote the first. Then I come with the other [? sine ?] plus the integral from 0 to a of k prime of x dx prime minus pi over 4. And let's call this thing delta.

So let's explore this wave function a little more. So things have become kind of nice. There's no x dependence here. That's very nice about this part of the formula. This is an angle. No x dependence. And the x dependence is just here from this term. And it's a nice x dependence because it's an integral from 0 to x. So it's kind of nice. The upper limit has the x. It's all kind of elegant.

So trigonometric function of this cosine of a difference of things is equal to its cosine of the first term, 0 to x k of x prime dx prime cosine delta plus sine of the first term 0 to x k of x prime dx prime sine delta. You have this quantity and this delta. So I use this trigonometric sum.

OK. But now you can see something nice. What did we say about the wave function? It had to vanish at the origin. And let's look at these two functions. Which one vanishes at the origin? You have cosine of the integral from 0 to x when x is equal to 0 at the origin is cosine of 0, which is 1. On the other hand, when x is small, x goes to 0. You get 0 for the integral and the sine vanishes. So this is the right term.

So this wave function would be correct if this term would be absent. This is a term that you must make absent. Psi of x without that term would be a valid wave function. It is the wave function. Therefore, the right wave function if we demand that cosine delta be equal to 0. So now we're imposing a very non-trivial condition. This wave over there. This contribution to the argument, to the angle here, this delta must be such that cosine delta is 0. In which case, this term would disappear and we would have a good wave function. Psi of x, if this is true, is 1 over square root of k of x times sine times the sine delta, which is another number. Sine of 0 to x k of x prime dx prime.

So we need cosine delta equal to 0. You know, this argument gives you this wave function in a nice way here written what it is. But we would have been able to find this even faster. If you just demand that Psi at x equals 0 is 0, you must have that this thing, the integral from 0 to a of this quantity minus pi over 4 must have 0 cosine. And that's the condition we did fine.

The advantage of our rearrangement is that when that happens, the whole wave function looks like that. And that's kind of nice. That gives you the picture of the wave function. A fairly accurate picture of the wave function in this region. Not very near the turning point, but you got that.

So what is this cosine delta going to 0? Well then delta must be 2n plus 1 times pi over 2. So the places where the cosine is 0 is pi over 2. That's for n equals 0. 3 pi over 2. So for n equal 1. And it just goes. The vertical axis and the unit circle. So this is for n equals 0, 1, 2, 3, goes on and on.

And this is a very wonderful condition. This says that the integral from 0 to a of k of x prime dx prime minus pi over 4. So actually, we have here you can multiply the 2 here. You get an n. And then you have pi over 2. And the pi over 4 that comes from that term becomes n plus 3/4 pi. It's a Bohr-Sommerfeld quintessential condition.

Look, I box that equation because that really gives you the answer. Now, why? Because you know what k is. k p of x is equal to h bar k is square root of 2n E minus v of x. And if you know E, you know a. So you have now an integral. And you take, for example, n equals 0. So you want the integral to be equal to 3/4 pi. But you will have to start changing the value of the energy with your computer if you cannot do the integral analytically and find, oops, for this value of the energy the integral of what I know, this is a known function, gives me this value.

So this is a very practical way of finding the energy levels. It does give you the approximate energy levels. And it's remarkably accurate in many cases. It also has a little intuition this is for n equals 0 would be like the ground state. n equal 1 first take on excited states. And indeed, that makes sense. Because this integral from 0 to a of k is the total phase of the wave function as you move from 0 to a.

And that wave is a number of factor n times pi plus a little bit. So it has time for m 0s. The phase as you move from 0 to a in the wave function, the sine function will have n 0s if this condition is satisfied consistent with that solution. So that's it. We'll continue next time. Find the solution of WKB exactly. All right.