

PROFESSOR: This also solves the exchange degeneracy problem. It kind of seems clear that it solves it. But since we made such a fuss about it, let's see it in more detail. So in what sense it solves it?

It solves it in the sense that you basically don't get degeneracy once you have this postulate. Before this postulate, for example, you had the two electron state. You said, well, it could be this, or it could be that. And which should I choose? If they're equivalent, we get into mistakes. So which one is the state of the two electrons?

Now we seem to have an answer. It's the antisymmetric combination. The last part of the answer is to say, well, there should be just one antisymmetric combination. And when you have two particles, it's kind of obvious. But if you have more, you can think a little. And you will conclude that it's kind of obvious as well. But let's just check it.

So this would be comment number 5, which is solving the exchange degeneracy problem. So this is the way we solve this problem, we discuss it. Suppose we have a state u that belongs to V tensor N . Now I should be a little tongue tied when I say this thing, a state u belonging to this. This is not a physical state. We already declared that arbitrary states in $V \otimes N$ are not physical states.

What should they call this? Maybe a mathematical state or a state, and then the others, call them physical states. You get the point. Assume you have this u that belongs here.

Then you can define V_u , a vector space associated with u , which is the span, all you get by applying permutations, the span of $P \alpha u$ for all $P \alpha$ in the permutation group S_N . You see, that was the idea of the degeneracy exchange. You got one state. And you produce all these other ones by permutation. And then you don't know which one to use. So we're going back to that situation.

You have one state. You produce all you can by perturbations. And now you've got a big vector space. And which one to use? Well, you know that you should use the states that are totally symmetric or antisymmetric. But let's first just get the idea clear of what we have here.

Notice, that depending on u , which u you choose, this vector space may be 1-dimensional, 2-dimensional, 3-dimensional, very large dimension. Could be of arbitrarily high dimension, up to n factorial.

So let's take an example. Suppose u is the state $a a b$ with tensor N equal 3. And you choose this. Well, if you apply permutation operators, you can also get the state $b a a$. You flip the first and the third. Or you can get $a b a$. And that's it. So in this case, you get three states in $V u$, for example.

And here is the claim. The claim is that there's just one state that is symmetric in $V u$. Up claim, up to scale $V u$ contains at most a single ket in $\text{Sym } N V$ and a single ket in $\text{Anti } N V$. That is the claim that we have solved the problem.

Now there's one state. You produce the whole list of states by permutation. There's just one. I said at most, because sometimes in the case of Anti-- I should say here actually, up the scale $V u$ contains a single ket and at most a single ket in $\text{Anti } N V$.

For example, the one we did up there, that example. If you wanted to produce a totally antisymmetric state, you would not be able to do it because of the a here. You cannot and antisymmetrize with states a and a . If you have a state of three fermions, they must each be different. And then you antisymmetrize. You cannot antisymmetrize states that are the same. So that's the statement here.

Let's say a couple of words about it. So the first thing we can say is that, yes, this is clear. You can get a state in $\text{Sym } N V$ by doing the symmetrizer acting in u . So it's clear you can get one state. And you can get a state in $\text{Anti } N$ on V by applying the antisymmetrizer on u . And you know. You can get those states directly.

Now, it may happen that Anti of V vanishes, because in Anti , when you apply a , you have some permutations with plus, some with minus, and there could be a cancellation. In this case, there would be a cancellation. In the symmetrize case, there's no cancellation. Possible So we'll see that.

So here it is. The argument for not getting more is kind of simple. Let me go through it. It's almost kind of obvious. You could say, OK, I got one state applying the projector. So that's it, there's one state.

Well, you could say, well, maybe somebody hands you another state. And then you would claim, oh, it must be the same as this one. So let's see that that is the case.

Suppose ψ is also in $\text{Sym } N V$ and belongs to $V u$. So it would be a claim that there's another

state that is symmetric. So since it belongs to V_u , ψ would be the sum of C_α times P_α on u . What was V_u was the span of this state. So if it's in that vector space, it would have to be in that space.

But since ψ is symmetric, ψ is also equal to $S\psi$ by assumption or S acting on $C_\alpha P_\alpha u$, which is ψ . But the S comes in and multiplies the P_α . And you'll recall that S times P_α is S . So you get this. How nice.

If you get that, this is a state that is independent of α . So you get it out here, $S u$ times sum over α of C_α . So what have you proven? If you hand out another state that you claim is in the symmetric subspace, it is proportional to the one you obtain with the projector. There's no other state you can find. You're done with these things.

So I want to make yet another comment, comment 6 and comment 7, and we'll be done.

Comment 6 is a construction for fermions. It's a famous construction. 6, for fermions, suppose you have a state u this time, which is ϕ on 1, χ on the second, and ω on the third. This time, we have to build an antisymmetric state. So the state A on u , the antisymmetric projector, is $\frac{1}{6} \sum_{\alpha} E_\alpha$ times the permutation P_α acting on this state.

So this will be a sum of six terms in which these states are scrambled, sometimes with pluses and sometimes with minuses. There is a nice construction in mathematics that does that. Moreover, you find that if you have two states that are the same, if ϕ would be equal to χ , you cannot antisymmetrize here. You will get 0. This is kind of like a statement that the two columns of a matrix have 0 determinant if they're proportional to each other.

So the claim is that this construction is nothing else than $\frac{1}{3!}$ the following determinant in which the three states are written like that, $\phi, \chi, \omega, \phi, \chi, \omega$, and ϕ, χ, ω into the determinant. And then you'll put 1, 2, 3, 1, 2, 3, and 1, 2, 3. The determinant of the states written in that way--

I would recommend any one to just write it out. For three particles, you have all the six elements. Write them out. Do the permutations. And check that this product produces that. You know, for example, that the determinant will produce this product along the diagonal with a plus sign, this and that with a plus sign, this and that with a plus sign, then the other ones with a minus sign, but each one 3. So for example, this, this, that is the one in which you have the identity acting on the states.

And the reason this works is the general formula. I will put it in the notes. But here, I think I don't want to get into more of it. It's a general formula for determinants. It's a well known formula.

If you have a permutation P , remember permutation P , α was supposed to be a list, α of 1, α of 2, up to α of n , which is a permutation of the list 1, 1, up to n . So if you have a matrix, the determinant of a matrix B , of any matrix, can be written as the sum over permutations of the sign of the permutation times, essentially, the permutation acting on the elements of the matrix, that is B of α 1 1, B of α 2 2, up to B of α n n .

This is like saying in the matrix, you will select the α 1 element from the first column, the α 2 element from the second column, all of those, and multiply them. And that's the determinant. And that formula is the reason this is true, because this sum of epsilons times P s really is the formula that calculates the determinant.

You almost see the P here. The P is the thing that took the element 1, 1, to 2, 2 and and n , n and replaced the 1 by α of 1, P of 2 and replaced them that way. So I almost made the connection. But we probably need a few minutes to make sure you understand why it happened.

Last comment, seventh, is occupation numbers. If we're trying to describe states in V tensor n , and suppose you have a state with n particles, and that is V tensor n . But suppose that this vector space V has a basis of basis vectors u 1, u 2, up to u infinity. It just goes on forever. Then if you have that, then there is a way to think of all the states that you can build. And the way to think of them is follows.

List the vectors here, u 2 up to all those. And you imagine that little n 1 particles are in this state. Little n 2 particles are in this state. And it goes on like that.

So you write your state with occupation numbers, n 1, n 2, n 3. And what is that state? Well, there's n 1 of this one. So u 1 for the first particle up to u 1 for the and n 1 particle. And then you have u 2 for the next particle up to a u 2. And you have n 2 of those. And you have all the others.

And then you must apply into it the symmetrization operator. But this is the list of states. This describes what you have. Symmetrizing is a lot of work. But listing the occupation numbers is really all there is to it, because the rest is a complicated algebraic manipulation of

symmetrization. But now you know what you must symmetrize.

You have n_1 times the state 1, n_2 times the state 2. When you have fermions, this is the same, except that the n_1 s, n_2 s, n_3 s can be 0 or 1. And finally here, if it's a state of n particles, then $n_1 + n_2 + \dots$ all of those must add up to n , because you have n particles.

So that brings us to the end of 8.06. Occupation numbers is where quantum field theory begins. You start constructing operators that add particles and subtract particles. So it's a fitting place to end your undergrad with quantum mechanics education.

I wish you all the best. I hope the course lived up to your expectations. It's an awesome responsibility to teach such an interesting course. And we try to do it well. And we wish that all of you have enjoyed this and have been motivated. And many of you will understand this subject, one day, better than your teachers and will help move it forward. So take care. Good luck. And see you soon. Bye.

[APPLAUSE]