PROFESSOR: Two particles is interesting in many cases. But in order to see what really is happening and how much structure you have to go to more than two particles. Two particles is a little too special. So we need to go beyond two particles and see what happens with this operator. So I'll do that. So we'll add more particles.

So let's do that. So if you have n particles, n particles, capital n, we can consider also permutations. And how many permutations are you going to have? You're going to have n factorial permutations. And notation of Hilbert space, n factorial permutation operators. So why you say n factorial? Because a permutation is a reordering of your n objects. And therefore, in order to define the reordering, you must decide which object is going to be the new first.

You have your n objects there. So you have n choices to pick a first and n minus 1 choices to be a second, n minus 2. So n factorial is the number of permutations. The permutations form a group. So the permutations of n objects form this symmetric group s capital n with n factorial elements. Just like you study the group su2 of two by two unitary matrices, or the group so3 of three dimensional rotations in space. The permutation group is the group that deserves a lot of study.

It's a finite discrete group. It's not a continuous group of transformations. It's discrete transformations. And they do funny things. So we need to understand this group so let's describe the notation of what is a permutation. Suppose you have n equal 4, four particles. So a given state is represented by a, b, c, d. And you could say this is the first particle, second, third, and fourth. And these are the states.

And now we're going to write the perturbation. The best thing, I think, is just do an example, p3142. So a permutation now has capital n labels, so four labels. And this is a permutation acting on this state. Remember the 2-1 permutation exchange. The state in 2 was put in 1. The state in 1 was put in 2. Here is the instruction.

So the instruction is, put in position one the third state. So here it says 3, so the third state goes into the first position. So that's the first position. The first state goes into the second position. The fourth state goes into the third position. The third in this here. So the third position is now occupied by the fourth state, d. And the second state goes into the fourth position, b. So this is the way we are going to define permutations. People define them in different ways and you have to be sure you know what it means.

But I'll right this. This means put third object in first position. Here the second one, let's write put first object in second position. It's nice to have this notation because now you can understand how the group works. A group of transformations is a set of elements that can be multiplied. There is an identity. And things have inverses as well. So permutations have inverses. If you rearrange the objects in one way, you can rearrange them back.

So for example, what would be the inverse of this permutation? I will try to figure out what is the inverse of p3142. So I look at this thing and I think of what is the instruction that I must put in order to rearrange back this series. And I say, well, I must put the second back in position one. So there should be a p2 here. I should put the fourth in position two.

And then I should put the first in position three. First in position three. And I should put the third in position four. So I claim that is the inverse of this operator, p213 is the inverse of this. So you can check it. Again, p2413 times p3142, you just do it on a, b, c, d. Well, the first one, you already know, is 2413, is put the third in one. Put the first in two. Put the fourth in three. And put the second in fourth, b.

And then it says, put the second first-- so that's an a. Put the fourth, second, b. Put the first, third, c. And put the third, fourth, d. So it's back to the original one. So indeed, it is the inverse. So you can write your permutations and play with them and figure out how to multiply any two permutations. More generally, we'll write the permutation as follows.

Suppose you have more generally a permutations [INAUDIBLE] with some index alpha, collective index alpha, that says how the integers, 1 up to n, are rearranged into alpha of 1, alpha of 2, up to alpha of n. So think of a permutation and thinking n integers and rearranging them. So the first one, the new first index is going to be alpha of 1, alpha of 2. And alpha is a rule that takes the n integers from 1 to n and reshuffles them.

So if you have a state-- so this is a simple thing because then if you have a p alpha, this alpha means really p alpha1, alpha2, all these things. Be alpha1, alpha2, alpha3, alpha4. If those act on u1 1, u2 2, up to u n, capital n, n, what does it give you? Well, it says here that you should put the 1 that has alpha 1 here on the first position. So you should put, first, the u alpha1.

And then you should put next the u alpha2. And all the way up to the u alpha n. That's in formulas, the rule that I was giving you above there. It tells you how to move the state from one to the other. You see, if alpha1 was 3, it would correspond to take the state u3 and putting it first. So it's like that. So whatever alpha 1, is whatever number, that state must be pulled into the first [INAUDIBLE].

OK, so let's just think a little about an example because it starts getting interesting. So let's do three particles for a little while, just to see what it is. So three particles, n equal 3. So what are the permutation operators? You would say p123 is what? Put the first particle first, the second particle second, the third particle third. Doesn't do anything. That is the identity.

So that's the identity element. But then you could cycle them, p321, p231. These are two more elements. This should have six elements because it's 3 factorial. And now you have the following one, p132, for example. That was not in the list. You had the p123. Now you flip this to p132. If I cycle them, p321, and p213, these are it.

The middle one, 312, yes. I was supposed to be cycling them, 312 and 231, yes. Otherwise I would have repeated that. Good eye. OK, so these are our operators. And these ones are kind of funny. They are simpler ones. Please look at them. This operator says put the first particle in the first position. So the first particle is left unchanged.

But the 3 and 2 are permuted. So people sometimes write this. They say, this permutation operator is just a single transposition, 32 or 23. These two are transposed and the first is left invariant. So this can be written as a single transposition. In this permutation operator, a 2 is left invariant. The second state is copied to the second position. But the 1 and 3 are exchanged, so this is a transposition of 1 and 3.

And in this last permutation operator, the third state is left invariant on the third position. Nevertheless, the second is put in first and the first is put in second. So it's a 12 transposition. OK, so the group has broken into things that are transpositions and slightly more complicated things. Now, let's look at the transpositions. So these are transpositions.

OK, what about the transpositions? Transpositions are like the permutation operators we were doing before of two particles. You're just changing two particles. You're leaving the other ones unchanged. So these transpositions are Hermitian and unitary. Positions are Hermitian and unitary, just like we had p21 before.

Now here comes the first statement that is quite remarkable about this permutation group. Any permutation, now we claim, is the product of some transpositions. You see, if somebody gives you a permutation of n integers, I think you're convinced that by changing two at a time, you can reach that. So that means that with transpositions, you can reach every possible element.

So any-- this is very important-- any permutation is the product of transpositions. And that implies, in a sense, the main result that you want from operators in this Hilbert space. Since transpositions are unitary and any permutation is a product of transpositions, any permutation is unitary. So all permutations are unitary.

It's not true that all permutations are Hermitian. You see, when you have the product of her Hermitian operators the product of her Hermitian operators is not necessarily Hermitian. You've seen that. If you have two operators that are Hermitian, the order changes. And the product is not Hermitian, x and p. Hermitian operators, the product is not Hermitian. So all permutations are unitary but are not Hermitian.

And the third concept that comes here illustrated [INAUDIBLE] also a very fundamental concept. We said that any permutation is a product of transpositions. But it's not a unique product of transposition. In fact, you could-- if you are given some order of objects and you're asked to rearrange them by transposition, you may do them in different order. So the product is not uniquely determined.

The product is, in fact, you could find somebody that does a permutation with some number of transpositions and another person with another set of transpositions. But one thing that has to happen-- and it's always the same-- is that the number of transpositions is the same [INAUDIBLE] two. That is, if you find that to get a permutation you need an even number of transpositions, everybody else will need also an even number of transpositions.

If you find an odd number of transpositions, everybody will find an odd number of transpositions. And that's kind of clear if you have-- if you've achieved the desired order by doing some number of transpositions, you can add two more transpositions and not change anything. You can flip two objects and flip them back. So this is a pretty important result as well.

So it means that permutations are either even or are odd. So permutations are even, if built with an even number of transpositions or odd if built with an odd number of transpositions. So two kinds of permutations, even permutations or odd permutations. So all these things we've kind of seen with this permutation group.

There's one more thing that is necessary to note. Something that may have seemed that coincidence there. In fact, the 3 on the right are transpositions. And they're a single transposition each. So each one of those on the right is an odd permutation. It's built with one transposition. On the other hand, the other ones are even transpositions, the first three. The first one is built with zero transpositions but the others are built with two transpositions.

It's something you could just see in detail. In fact, I would say it's worth doing the table of multiplication of this group. But that coincidence is, in fact, true in any-- in sn, in the group sn, the number of even permutations is equal to the number of odd permutations.

Some pretty nice result. There's equal numbers of even and odd permutations in any permutation group. Actually, it's surprisingly simple to show that. You think of the even permutations here-- here are the even permutations. And here are the odd permutations. And you can easily show that there is equal number on this one. So let do it.

One way is to take the permutation element. So suppose you have a permutation-- how should I call it-- p21. You leave everything the same but you have p21. That permutes the second and first labels. Since it's an n group, the rest are left invariant, so 3, 4, 5, 6, 7, 8, 9, 10. And you say, this maps any permutation here into a permutation there. Why?

Because if you have an even permutation here, any element here has an even number of transient mutations, of transpositions. So this is another transposition, it transposes 2 and 1. So when you multiply an element here by p21, if it has an even number of transpositions, now it has an odd one. So it must land somewhere here.

So any-- this p21 maps the even-- the even permutations to the odd permutations. Moreover, it maps them in what's called 1 to 1 fashion. If you have two permutations here that are different, it will map to two permutations here that are different. Why? Because suppose you have here pi and pj, two permutations. 1 is mapped to p21 pi, and the other is map to p21 pj.

But if pi and pj are different, two permutations here, I and j, pi and pj. And if they are different, then it follows that these two are different as well after you multiply them by p21. Why? Because if they were equal, p21 has an inverse. It is itself p21, so you could multiply them in. And then you would show that pi is equal to pj. And that can't happen.

So two different elements here will go to two different elements there. So if you think of all your elements here, go-- if you have 20 elements here, they go to 20 elements here. The only thing that can happen is that some elements are not reached here. That would mean that these two sets are not the same. So you need to show that this map reaches everybody.

So this shows it's a one to one map. And in fact, it's kind of a joke, but in a way to think of this, one to one really means two to two, in the sense that two things go to two things. So it's kind of a funny name, one to one. So the other thing is that you need to show that everybody reaches. So the map is-- let's call surjective, everybody reaches. So it's clear that everybody reaches because if you have a pk in here, permutation pk, this pk can be written as p21 times p21 times pk. Because p21 times p21 is 1.

And this pk, since this was an odd permutation, this is an even one. And then you've shown that pk is obtained by acting with p21 on some even permutation. So you'll reach. So the map is surjective. And therefore these two things are the same. They're identical numbers of even and odd permutations, a very nice fact about these groups.