PROFESSOR: Let's then try to understand intuitively some issues with phase shift. And that, I think, is pretty valuable. One shouldn't rush through these things. I'm going to erase this formula already. It's a little messy. It's not like you're going to use it ever. You could use that formula if you were to compute higher phase shifts. But at least this homework, you don't have that.

So let's try to understand a little what phase shifts do for you. Phase shifts are useful when the first few phase shifts dominate the cross section. If you have to do this sum for all values of l, it can get very difficult. So if the answer really necessitates the tail of l going to infinity, it's hard. So in general, we find that phase shifts are useful when the first few dominate the cross section. So phase shifts are useful if the first few dominate sigma.

And that will tend to happen when k a is much less than 1.

And this will happen in that case. That will be our argumentation. We will try to explain why when k a is less than 1, this will happen. So what is k a less than 1? Well, it could be that a is very small. So one possibility is short range, which means a small compared to other scales, and maybe there is a problem. Or the other possibility is that low energy. That is, k is small, because energy goes like k squared. If k is small, it's low energy.

So these are the possibilities that would allow us to have a situation where only the first few phase shifts contribute to the cross section. So let's see why that is the case and what is the intuition for this. This happens for-- I think I want to add here this happens when-- so the claim above happens when k a is less than 1.

Let's have a picture of this. Consider an incident particle on a potential. So here is a potential. Here is r equals 0 and this is the potential. Maybe some region-- here's an incident particle coming in with some momentum. So these ideas are semi-classical ideas on how I'm looking at this particle coming with some impact parameter. b is this distance to the center, the maximum-- the closest approach of the particle to the potential. That's called the impact parameter. And it has momentum p. So b is the impact parameter and p is the momentum.

And then, what is the angular momentum of this particle with respect to the origin? Is r cross p. And so the angular momentum l is equal to bp.

OK. So now I'm going to try to make sense of this equation and get an estimate, considering our situation. So what is our situation? In our situation, when we consider a partial wave expansion, we're considering waves with different values of the orbital angular momentum l. So when we consider a little l, we're really saying that the angular momentum is proportional to hl, where l is the quantum number. Number. The number that we're fixing here each time we talk about the partial wave.

Moreover, if we're asking about the momentum, we know what is the typical momentum of these waves-- it's hk.

So if I substitute those two pieces of information into the relation of the angular momentum here-- so l is equal to hl and b multiplies h bar k, we get that the impact parameter can be visualized as l over k.

It's a somewhat classical intuition. But it gives you an idea. As you're increasing l, you're increasing the impact parameter intuitively. A partial wave scattering process with l reflects the input parameter. If l is equal to 0, you're hitting the object right on.

And now, you can say, all right, if b is greater than a, where a is the range of the potential-- range-- there should be little or no scattering. So if b is greater than a, is little or no scattering. That's intuitively clear from the range of the potential. If the input parameter is much bigger. So this means l greater than k a, there is no scattering. And this physically-- this is classical intrusion suggests that the contribution to scattering from l's bigger than k a is negligible. There's no scattering.

So this is saying that sigma-- how should I say it? The delta l-- I'll just say delta l's are small or very small-- small-- for l greater than k a. The delta l, the phase shifts, enter here, as sine squared of the phase shifts. So they must be very small, because they just pretty much don't contribute.

This is intuition we were claiming up there. If k a is rather small, very few phase shifts will contribute. Because as soon as your l is bigger than k a, you get nothing. If k a is much less than 1, only l equals 0 will contribute. If k a is 1 or 2, it's-- I think I would write here, if-- we don't have to be very small, but around 1, very few will contribute. Or essentially, you could say that all the l less than k contribute. They contribute. So even if k a, say, was 5, you could hope that the l's up to 5 will be a big contribution. And after that, they start to fall off.

OK. So that's one intuition into the l's. Second piece of intuition comes from the solutions that you have of the real equation. So it's interesting, actually. Let's look at our partial waves directly.

Second intuition for this fact-- think of your solution Jl of kr, Yl of theta. This is a solution. Represents a partial wave. It's the one that doesn't diverge at the origin. So that makes some sense. It represents a free wave, but still is giving us an intuition of how this partial wave is supposed to behave. So I would like to have an intuition of how big is this function and where it is big.

So this is a solution of the radial equation. A radial equation with no potential. That doesn't mean zero effective potential, because there's always the centrifugal barrier. So this is the solution. Solution of the v equals 0 radial equation, which has a potential, which is v effective is h squared, l times l plus 1 over 2mr squared. That's the effective potential. And it's a solution with energy h squared, k squared, over 2m.

So here is r. Here is the effective potential. And how does it look? Well, it diverges and then falls off. And you have a solution with energy equal h squared, k squared, over 2m. It's a solution of that problem with that potential. So here is your intuition.

This solution encounters a barrier here. So it must be exponentially suppressed in this distance. It will be exponentially suppressed as you go below this turning point. Therefore, this partial wave will be 0 for some radius smaller to a certain distance here. And that's back of our intuition. If you have some impact parameter this large, you have no support over distances smaller than your impact parameter. There's no wave there. The wave stays away.

So what we want to show is that this wave stays away from the center r equals 0, again, by a quantity related to this thing. So what do we get? Well, we can solve for the turning point. So h squared l, l plus 1 over 2mr star squared-- I can call this r star-- is equal to h squared, k squared, over 2m. Lots of things cancel. 2m's, h squared. So you get kr star squared is roughly equal to l times l plus 1. And here, I can approximate and I can say that kr star is roughly l. That's the leading order square root.

And therefore, the wave function vanishes. This Jl of kr must be quite approximately 0 for r less than r star equal l over k, which is the impact parameter.

So we know because this must be exponentially suppressed, that the wave function must be very much 0 here. And that the radial wave function is this. So you have derived that the Jl of kr is essentially 0 for r less than this r star, which is about l over k, that we identified after impact parameter. So our ideas are consistent. This partial wave doesn't reach the origin. It stays a bit the way. How much? The intuition is that the distance approximate to the input parameter, which is l divided by k.

OK. So that's another piece of intuition. There's one more classical intuition that they think it's interesting. It's very surprising, actually, this third one, this semi-classical intuition into what these things are. But they're important because the subject of phase shift seems very technical. But the phase shifts have important meaning and those partial waves are interesting. So let's do one more way of thinking about these matters.

OK. So I'd say, recall this equation. Sigma is given by the total cross section-- three-- the total cross section is given by this sum. So sigma is the sum of sigma l's. And that's a nice notation. This is the stigma associated with each partial wave. No, it's a good thinking. And the sigma l is equal to 4 pi over k squared, 2l plus 1, sine squared of delta l.

Now, just from this formula, when somebody gives you a sigma l, you can know whether they're making sense or not, because there is a test, at least. Sigma l is always less than or equal to 4 pi over k squared times 2l plus 1. And this just comes because sine squared is less than or equal to 1. So it's a good test. People say, oh, I found the partial cross section given by this for that partial wave. Well, it better satisfy that, otherwise, you've done a wrong calculation.

Some people call this partial wave unitarity. Unitarity has to do with conservation of probability and cross sections have that thing. It's a probability to scatter in some direction, in some measure. So this is called partial wave unitarity. Wave unitarity.

It's interesting that these ideas have a connection in classical physics, again, that allow you to think of, actually, physically about these processes in a way that illuminates what's happening quantum mechanically. So here is how you can think of this. We have, again-- I'll do a nice picture. I'll try at least. A source here and there's a potential. And now I'm going to come at some impact parameter. But let me think of it as coming-- here is the cylinder of impact parameters. If you come anywhere on the surface of this cylinder, you'll have input parameter b-- b is here.

Now, let's fit in the cylinder. I'll fit in the cylinder. To make it of thickness delta b. Thickness. So the thickness is delta b. And you now consider all these particles that are going to come in here and scatter.

Now, in classical physics, you know it's easy or reasonable to track things, because everything is deterministic. So I can imagine very well that this shell just goes here. And it becomes a shell here of particles that are going off at some angles. So, like a trumpet. This comes like that and spreads out here. And all these particles get scattered. Deterministically, that's reasonable. You have a charge here, for example, and these are charged particles. They repel and they get scattered like that.

So you could say, OK, this contribution is going to go into a particular angle, the omega. So I can think of this as contributing to the differential cross section at an angle omega, or some contribution to the total cross section that would come from integrating and adding up all these contributions as you vary this thing.

So if I want to compute how much this contributes to the cross section-- estimate the classical contribution. Estimate the classical contribution d sigma to sigma from a single partial wave-- single partial wave-- if I consider a partial wave with some value of l. Think of a partial wave with a fixed value of l. Then, I have fixed the impact parameter. If I change l by one unit, the impact parameter changes a little bit. Think of this as the b of a given l-- b of a given l. And this delta b or db-- I'll call it delta b with capital. Delta b. As they change in b for delta l equal 1.

You see, each partial wave corresponds to a different l. Classically, there's no quantization. So I must think of an impact parameter. And the partial wave corresponds to the thickness of this thing when the thickness will correspond to a change of b, because l has changed one. So you can think of concentric cylinders. The l equals 0 partial wave, the l equal 1 partial wave, all of those are cylinders. And they all contribute areas. This is the area that is going to get-- all the particles are going to scatter. So this little thin cross-sectional area here is the contribution to the differential cross section from this partial wave in the classical approximation.

So let's do this. So b of l is equal to l over k. And delta b is the change in b when l changes by 1. So delta b is 1 over k. So what is the differential cross section? The classical-- classical-- it's this area. This is the area of the beam that is going to be scattered in that partial wave. So it's going to be 2 pi b delta b. And this is 2 pi l over k times 1 over k. And that's not so bad.

This is-- I'll write it as 1 over 4, 4 pi over k squared-- the k squared came out right-- times 2l. I wrote it like that because-- here it is-- what the quantum theory tells you the partial wave contribution to the cross section is 4 pi over k squared 2l plus 1. That's pretty much it. And the classical estimate gives you one fourth of that.

So pretty nice that everything works out. And that there is a correspondence. And there is a way to estimate partial contributions to the cross section from classical arguments. So that's our discussion of the physics of phase shifts and the physics of input parameters and I and your intuition that you must have to them.