MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Solutions, Practice Exam #2

Problem 1 (35 points) Weakly Interacting Bose Gas

a)

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV$$

$$\left(\frac{\partial P}{\partial V}\right)_T = -2cV^{-3} \quad \text{from given}$$

$$\left(\frac{\partial P}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T = \frac{5}{2}aT^{3/2} \quad \text{from a Maxwell relation and the given}$$

$$dP = \frac{5}{2}aT^{3/2}\,dT - 2cV^{-3}\,dV$$

$$P = aT^{5/2} + f(V)$$

$$\left(\frac{\partial P}{\partial V}\right)_T = f'(V) = -2cV^{-3} \quad \Rightarrow \quad f(V) = cV^{-2} + \underbrace{(0)}_{\text{from limiting behavior}}$$

$$P(T,V) = \underline{aT^{5/2} + cV^{-2}}$$

b)

$$dU = TdS - PdV = T\left(\frac{\partial S}{\partial T}\right)_V dT + \left(T\left(\frac{\partial S}{\partial V}\right)_T - P\right)dV$$

$$= \frac{15}{4}aT^{3/2}V\,dT + \left(\frac{3}{2}aT^{5/2} - cV^{-2}\right)dV$$

$$U = \frac{3}{2}aT^{5/2}V + g(V), \quad \left(\frac{\partial U}{\partial V}\right)_T = \frac{3}{2}aT^{5/2} + g'(v)$$

$$g'(V) = cV^{-2} \quad \Rightarrow \quad g(V) = cV^{-1} + U_0$$

$$U(T,V) = \underline{\frac{3}{2}aT^{5/2}V + cV^{-1} + U_0}$$

c) The model obeys the 3^{rd} law because $\lim_{T\to 0} S(T,V) = 0$.

Problem 2 (30 points) Carnot heat engine

a) For a reversible process $đQ = TdS$. Also, for a constant volume process, $dQ = C_V dT$. Thus

$$\begin{aligned} dS &= \frac{dQ}{T} \\ \Delta S &= \int_{T_2}^{T_1} \frac{dQ}{T} = \int_{T_2}^{T_1} \frac{C_V}{T}\, dT \\ &= C_V \ln T_1/T_2 = \underline{-C_V \ln T_2/T_1} \end{aligned}$$

b) The efficiency of an engine cycle η is defined as (work out)/(heat extracted at the higher temperature). Thus

$$\begin{aligned} dW_{\text{out}} &= \eta\, |dQ_2| \\ &= \left(1 - \frac{T_1}{T_2}\right)(-C_V\, dT_2) \quad \text{for a Carnot cycle} \\ \Delta W_{\text{out } 2\to1} &= -\int_{T_2}^{T_1} \left(1 - \frac{T_1}{T_2}\right)(C_V\, dT_2) \\ &= -C_V\left[(T_1 - T_2) - T_1 \ln(T_1/T_2)\right] \\ &= \underline{C_V\left[(T_2 - T_1) - T_1 \ln(T_2/T_1)\right]} \end{aligned}$$

c) Since the engine is run in cycles and entropy is a state function, the entropy change in each cycle is zero, as is the the total entropy change in the process.

One can see this as well by applying conservation of energy.

Heat out at high T - Heat dumped at low T = Work out

$$\begin{aligned} \Delta Q_1 &= C_V(T_2 - T_1) - \Delta W_{\text{out } 2\to1} \\ &= T_1 \ln(T_2/T_1) \\ \Delta S_1 &= \Delta Q_1/T_1 = C_V \ln(T_2/T_1) = -\Delta S_2 \quad \text{found in a) above} \end{aligned}$$

Problem 3 (35 points) A Classical Ultra-relativistic Gas

a) For one atom

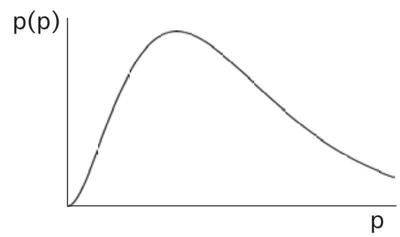
$$\begin{aligned} Z_1 &= \int \exp[-\epsilon/k_B T] dp^3 dV/h^3 \\ &= \frac{V}{h^3}\int_0^\infty \exp[-cp/kT]p^2\,dp \underbrace{\int_0^\pi \sin\theta\,d\theta \int_0^{2\pi} d\phi}_{4\pi} \\ &= 4\pi\frac{V}{h^3}\left(\frac{k_b T}{c}\right)^3 \underbrace{\int_0^\infty \exp[-y]y^2\,dy}_{2} \\ &= \underline{8\pi V\left(\frac{k_b T}{hc}\right)^3} \end{aligned}$$

For the whole gas

$$\underline{Z = \frac{1}{N!}z_1^N}$$

b) $p(p) \propto \exp[-cp/k_B T]\,p^2$. The normalization integral was done in a).

$$\underline{p(p) = \frac{1}{2}\left(\frac{c}{k_b T}\right)^3 \exp[-cp/k_B T]\,p^2 \quad \text{for } p \geq 0}$$



c) Note $Z = A\beta^{-3N}$.

$$U = -\frac{1}{Z}\left(\frac{\partial Z}{\partial \beta}\right)_V = (-\frac{1}{Z})(-3N)\frac{Z}{\beta} = \underline{3Nk_B T}$$

d)

$$\begin{aligned} F &= -k_B T \ln Z \\ P &= -\left(\frac{\partial F}{\partial V}\right)_T = k_B T\left(\frac{\partial \ln Z}{\partial V}\right)_T = k_B T \underbrace{\frac{1}{Z}\left(\frac{\partial Z}{\partial V}\right)_T}_{N/V} = \underline{\frac{Nk_B T}{V}} \end{aligned}$$

e)

$$\begin{aligned} F &= U - TS \to S = (U-F)/T = 3Nk_B + k_B \ln Z \\ S(T,V,N) &= \underline{3Nk_B + k_B \ln\left\{\frac{1}{N!}\left[8\pi V\left(\frac{k_B T}{hc}\right)^3\right]^N\right\}} \end{aligned}$$

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