MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Practice Exam #2

Problem 1 (35 points) Weakly Interacting Bose Gas

At low temperatures the entropy and isothermal compressibility of a weakly interacting Bose gas can be approximated by

$$S(T,V) = \frac{5}{2}\,aT^{3/2}\,V$$

$$\kappa_T \equiv -\frac{1}{V}\left.\frac{\partial V}{\partial P}\right|_T = \frac{1}{2c}\,V^2$$

where a and c are constants. In the limit of low temperature and high volume the pressure P and the internal energy density E/V approach zero.

a) (15) Find the equation of state $P(T,V)$.

b) (15) Find the internal energy $U(T,V)$.

c) (5) Does this model for the gas obey the third law of thermodynamics? Explain the reasoning behind your answer.

Problem 2 (30 points) Carnot heat engine

A reversible Carnot heat engine operates between two reservoirs with temperatures T_1 and T_2 where $T_2 > T_1$. The colder reservoir is so large that T_1 remains essentially constant. However, the hotter reservoir consists of a finite amount of ideal gas at constant volume, for which the heat capacity C_V is a given constant.

After the heat engine has run for some period of time, the temperature of the hotter reservoir is reduced from T_2 to T_1.

a) (10) What is the change in the entropy ΔS of the hotter reservoir during this period?

b) (10) How much work did the engine do during this period?

c) (10) What is the total change in the entropy of the system during this period?

Problem 3 (35 points) A Classical Ultra-relativistic Gas

A homogeneous gas of N classical, non-interacting, indistinguishable atoms is confined in a volume V. The gas is in thermal equilibrium at a temperature T which is so high that the energy of each atom can be approximated by its limiting ultra-relativistic limit:

$$\epsilon = cp \qquad \text{where } p \equiv |\vec{p}|$$

a) (7) Find the partition function for the gas, $Z(N, L, T)$. You may want to use spherical coordinates in which $dp^3 = p^2 \sin\theta \, dp d\theta d\phi$ where p is the magnitude of the momentum vector.

b) (7) Find the probability density for magnitude of the momentum, $p(p)$. Sketch the result.

c) (7) Find the internal energy of the gas, $U(T, V, N)$.

d) (7) Find the pressure, $P(T, V, N)$.

e) (7) Find the entropy, $S(T, V, N)$. [Hint: It is possible to do this without taking another derivative.]

Work in simple systems

Hydrostatic system	$-PdV$
Surface film	$\gamma\, dA$
Linear system	$\mathcal{F}dL$
Dielectric material	$\mathcal{E}d\mathcal{P}$
Magnetic material	HdM

Thermodynamic Potentials when work done on the system is $dW = Xdx$

Energy	E	$dE = TdS + Xdx$
Helmholtz free energy	$F = E - TS$	$dF = -SdT + Xdx$
Gibbs free energy	$G = E - TS - Xx$	$dG = -SdT - xdX$
Enthalpy	$H = E - Xx$	$dH = TdS - xdX$

Statistical Mechanics of a Quantum Harmonic Oscillator

$$\epsilon(n) = (n + \tfrac{1}{2})\hbar\omega \qquad n = 0, 1, 2, \ldots$$

$$p(n) = e^{-(n+\frac{1}{2})\hbar\omega/kT}/Z(T)$$

$$Z(T) = e^{-\frac{1}{2}\hbar\omega/kT}(1 - e^{-\hbar\omega/kT})^{-1}$$

$$< \epsilon(n) >= \tfrac{1}{2}\hbar\omega + \hbar\omega(e^{\hbar\omega/kT} - 1)^{-1}$$

Radiation laws

Kirchoff's law: $e(\omega, T)/\alpha(\omega, T) = \frac{1}{4}c\, u(\omega, T)$ for all materials where $e(\omega, T)$ is the emissive power and $\alpha(\omega, T)$ the absorptivity of the material and $u(\omega, T)$ is the universal blackbody energy density function.

Stefan-Boltzmann law: $e(T) = \sigma T^4$ for a blackbody where $e(T)$ is the emissive power integrated over all frequencies. ($\sigma = 56.9 \times 10^{-9}$ watt-m^{-2}K^{-4})

Integrals

$$\int e^{ax}\, dx = \frac{e^{ax}}{a}$$

$$\int x\, e^{ax}\, dx = \frac{e^{ax}}{a^2}(ax - 1)$$

$$\int x^2\, e^{ax}\, dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int \frac{dx}{1 + e^x} = \ln\left[\frac{e^x}{1 + e^x}\right]$$

Definite Integrals

For integer n and m

$$\int_0^\infty x^n e^{-x}\, dx = n!$$

$$\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx = \sqrt{\pi}$$

$$(2\pi\sigma^2)^{-1/2} \int_{-\infty}^\infty x^{2n} e^{-x^2/2\sigma^2}\, dx = 1 \cdot 3 \cdot 5 \cdots (2n - 1)\, \sigma^n$$

$$\int_0^\infty x\, e^{-x^2}\, dx = \frac{1}{2}$$

$$\int_0^1 x^m (1 - x)^n\, dx = \frac{n!m!}{(m + n + 1)!}$$

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