MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

Solutions, Exam #1

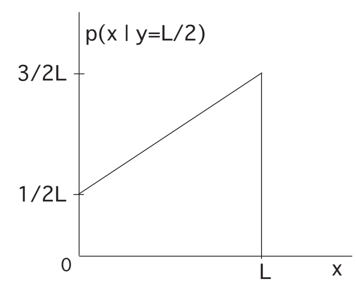
Problem 1 (30 points) Quantum Dots

a) Use Bayes' theorem: $p(x|y) = p(x,y)/p(y)$. We are given $p(x,y)$ so we must first find $p(y)$.

$$p(y) = \int p(x,y)\,dx = \frac{1}{L^3}\int_0^L (x+y)\,dx$$

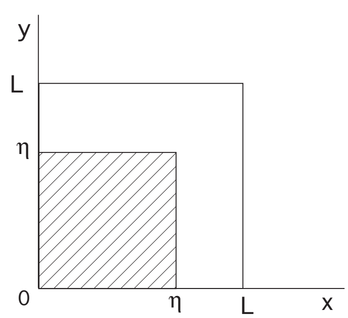
$$= \frac{1}{L^3}(L^2/2 + yL) = \frac{1}{L}(1/2 + y/L)$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{(x+y)/L^3}{(1/2+y/L)/L} = \underline{\frac{1}{L}\frac{(x/L+y/L)}{(1/2+y/L)}}$$



<u>x and y are not statistically independent.</u> You could either point out that $p(x,y) \neq p(x)p(y)$ or that $p(x|y)$ depends on y.

b)



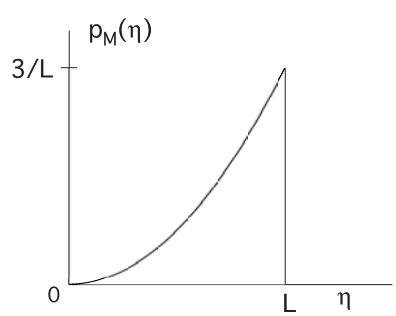
$M \leq \eta$ in the shaded region in the figure to the left.

$$P_M(\eta) = \frac{1}{L^3}\int_0^\eta \left(\int_0^\eta (x+y)\,dy\right) dx$$

$$= \frac{1}{L^3}\int_0^\eta \left(\eta x + \eta^2/2\right) dx$$

$$= \frac{1}{L^3}\left(\eta^3/2 + \eta^3/2\right) = \frac{\eta^3}{L^3}$$

$$p_M(\eta) = \frac{dP_M(\eta)}{d\eta} = \underline{\frac{3}{L}(\eta/L)^2} \quad \text{for} \quad 0 \leq \eta \leq L$$



Problem 2 (30 points) A Real Gas

Use the first law, solve for $đQ$, expand dU, set $đQ = 0$.

$$
\begin{aligned}
dU &= đQ + đW = đQ - PdV \\
đQ &= dU + PdV \\
&= \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\left(\frac{\partial U}{\partial V}\right)_T + P\right) dV \\
&= (2bV^{2/3}T)\,dT + (-(2/3)aV^{-5/3} + (2/3)bV^{-1/3}T^2 + (2/3)aV^{-5/3} + (2/3)bV^{-1/3}T^2)\,dV \\
&= (2bV^{2/3}T)\,dT + (\ (4/3)bV^{-1/3}T^2\ )\,dV = 0 \\
(2/3)T^2\,dV &= -VT\,dT \to \frac{dV}{V} = -(3/2)\frac{dT}{T} \to \ln(V/V_0) = -(3/2)\ln(T/T_0) \\
\frac{V}{V_0} &= \underline{\left(\frac{T}{T_0}\right)^{-3/2}}
\end{aligned}
$$

Problem 3 (40 points) Ultra-relativistic Gas in One Dimension

a)

$$\begin{aligned}
\Phi &= (\text{number of choices for s})^N \times \left[\int_0^L dx\right]^N \times \int_{\sum p_i \le E/c} dp_1 dp_2 \cdots dp_N \times \frac{1}{\hbar^N N!} \\
&= 2^N \times L^N \times \frac{1}{N!}\left(\frac{E}{c}\right)^N \times \frac{1}{\hbar^N N!} \\
&= \left(\frac{1}{N!}\right)^2 \left(\frac{2LE}{\hbar c}\right)^N \\
\Omega &= \Delta \left(\frac{\partial \Phi}{\partial E}\right)_{N,L} = \underline{\left(\frac{1}{N!}\right)^2 \left(\frac{2LE}{\hbar c}\right)^N \left(\frac{N\Delta}{E}\right)}
\end{aligned}$$

b)

$$dE = TdS + \mathcal{F}dL \quad \rightarrow \quad dS = \frac{1}{T}dE - \frac{\mathcal{F}}{T}dL$$

$$\begin{aligned}
S &= k_B \ln \Omega \\
&= k_B \left[N \ln\left(\frac{2LE}{\hbar c}\right) - 2\ln N!\right] \approx k_B \left[N \ln\left(\frac{2LE}{\hbar c}\right) - 2N\ln N + 2N\right] \\
&= Nk_B \left[\ln\left(\frac{2}{\hbar c}\frac{L}{N}\frac{E}{N}\right) + 2\right] \quad \text{which is properly extensive} \\
\left(\frac{\partial S}{\partial E}\right)_L &= \frac{1}{T} = Nk_B \frac{1}{()}\frac{()}{E} \quad \Rightarrow \quad \underline{E = Nk_BT}
\end{aligned}$$

c)

$$\left(\frac{\partial S}{\partial L}\right)_E = -\frac{\mathcal{F}}{T} = Nk_B \frac{1}{()}\frac{()}{L} \quad \Rightarrow \quad \underline{\mathcal{F} = -Nk_BT/L}$$

d) In b) you found an expression for the entropy of the gas. The entropy will be constant when the product LE is constant. Replacing E with the expression found later in b) gives LT is constant on any adiabatic path. Therefore the adiabat passing through the point (T_0, L_0) is given by

$$\underline{\frac{L(T)}{L_0} = \left(\frac{T}{T_0}\right)^{-1}}$$

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