MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Physics Department

8.044 Statistical Physics I — Spring Term 2013

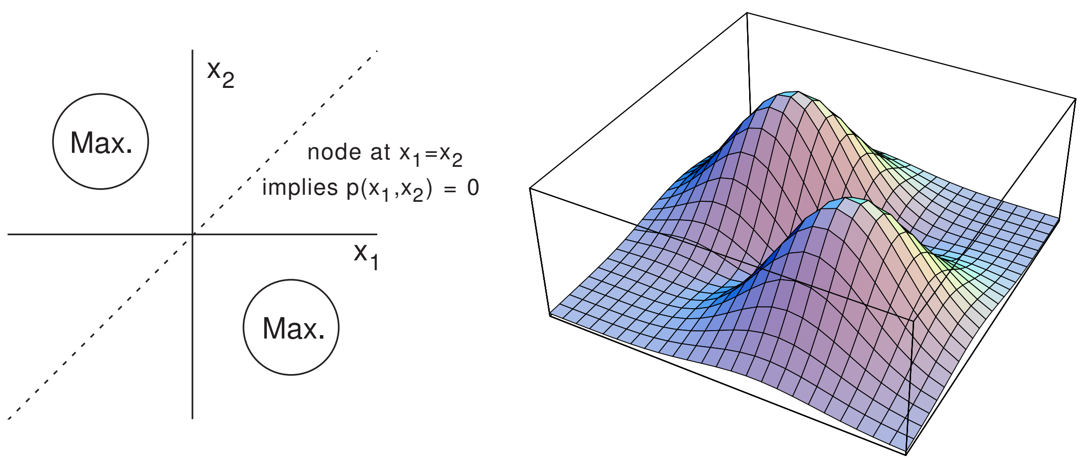
Solutions to Problem Set #2

Problem 1: Two Quantum Particles

a)

$$
\begin{aligned}
p(x_1, x_2) &= |\psi(x_1, x_2)|^2 \\
&= \underline{\frac{1}{\pi x_0^2}(\frac{x_1 - x_2}{x_0})^2 \exp[-\frac{x_1^2 + x_2^2}{x_0^2}]}
\end{aligned}
$$

The figure on the left shows in a simple way the location of the maxima and minima of this probability density. On the right is a plot generated by a computer application, in this case Mathematica.



b)

$$
\begin{aligned}
p(x_1) &= \int_{-\infty}^{\infty} p(x_1, x_2)\, dx_2 \\
&= \frac{1}{\pi x_0^2}\frac{1}{x_0^2}\exp[-x_1^2/x_0^2]\int_{-\infty}^{\infty}(x_1^2 - 2x_1x_2 + x_2^2)\exp[-x_2^2/x_0^2]\, dx_2 \\
&= \frac{1}{\pi x_0^2}\frac{1}{x_0^2}\exp[-x_1^2/x_0^2]\left[x_1^2(\sqrt{\pi x_0^2}\,) - 2x_1 \times 0 + \frac{x_0^2}{2}(\sqrt{\pi x_0^2}\,)\right] \\
&= \underline{\frac{1}{\sqrt{\pi x_0^2}}(x_1^2/x_0^2 + 1/2)\exp[-x_1^2/x_0^2]}
\end{aligned}
$$

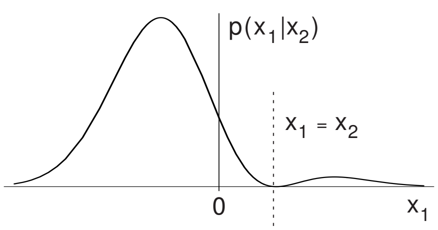
By symmetry, the result for $p(x_2)$ has the same functional form.

$$\underline{p(x_2) = \frac{1}{\sqrt{\pi x_0^2}}(x_2^2/x_0^2 + 1/2)\exp[-x_2^2/x_0^2]}$$

By inspection of these two results one sees that $p(x_1, x_2) \neq p(x_1)p(x_2)$, therefore $\underline{x_1 \text{ and } x_2 \text{ are not statistically independent}}$.

c)

$$\begin{aligned} p(x_1|x_2) &= p(x_1, x_2)/p(x_2) \\ &= \frac{\sqrt{\pi x_0^2}}{\pi x_0^2}\frac{(x_1 - x_2)^2}{(x_2^2 + \frac{1}{2}x_0^2)}\frac{\exp[-(x_1^2 + x_2^2)/x_0^2]}{\exp[-x_2^2/x_0^2]} \\ &= \underline{\frac{2}{\sqrt{\pi x_0^2}}\frac{1}{(1 + 2(x_2/x_0)^2)}\left(\frac{x_1 - x_2}{x_0}\right)^2\exp[-x_1^2/x_0^2]} \end{aligned}$$



It appears that these particles are anti-social: they avoid each other.

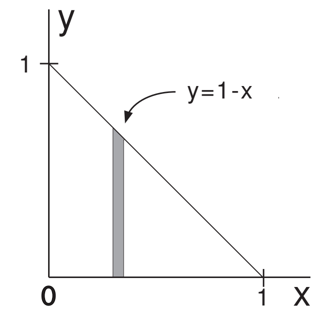
For those who have had some quantum mechanics, the $\psi(x_1, x_2)$ given here corresponds to two non-interacting spinless[1] Fermi particles (particles which obey Fermi-Dirac statistics) in a harmonic oscillator potential. The ground and first excited single particle states are used to construct the two-particle wavefunction. The wavefunction is antisymmetric in that it changes sign when the two particles are exchanged: $\psi(x_2, x_1) = -\psi(x_1, x_2)$. Note that this antisymmetric property precludes putting both particles in the same single particle state, for example both in the single particle ground state.

For spinless particles obeying Bose-Einstein statistics (Bosons) the wavefunction must be symmetric under interchange of the two particles: $\psi(x_2, x_1) = \psi(x_1, x_2)$. We can make such a wavefunction by replacing the term $x_1 - x_2$ in the current wavefunction by $x_1 + x_2$. Under these circumstances $p(x_1|x_2)$ could be substantial near $x_1 = x_2$.

[1] Why spinless? If the particles have spin, there is a spin part to the wavefunction. Under these circumstances the spin part of the wavefunction could carry the antisymmetry (assuming that the spatial and spin parts factor) and the spatial part of Fermionic wavefunction would have to be symmetric.

Problem 2: Pyramidal Density

a) The tricky part here is getting correct limits on the integral that must be done to eliminate y from the probability density. It is clear that the integral must start at $y = 0$, but one must also be careful to get the correct upper limit. A simple sketch such as that at the right is helpful.

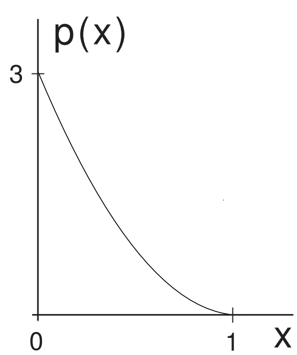


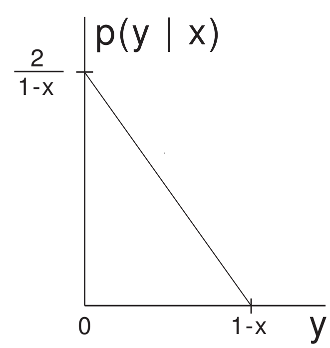
$$\begin{aligned} p(x) &= \int_{-\infty}^{\infty} p(x,y)\,dy = 6\int_0^{1-x}(1-x-y)\,dy \\ &= 6\{\left[(1-x)y - \frac{1}{2}y^2\right]_0^{1-x}\} = 6\{(1-x)^2 - \frac{1}{2}(1-x)^2\} \\ &= \underline{3(1-x)^2} \qquad 0 \le x \le 1 \end{aligned}$$

b)

$$\begin{aligned} p(y|x) &= \frac{p(x,y)}{p(x)} = \frac{6(1-x-y)}{3(1-x)^2} \\ &= \underline{\frac{2}{(1-x)^2}\,(1-x-y)} \qquad 0 \le y \le 1-x \end{aligned}$$

Note that this is simply a linear function of y.





Problem 3: Stars

a) First consider the quantity p(no stars in a sphere of radius r). Since the stars are distributed at random with a mean density ρ one can treat the problem as a Poisson process in three dimensions with the mean number of stars in the volume V given by $< n >= \rho V = \frac{4}{3}\pi\rho r^3$. Thus

$$\begin{aligned} p(\text{no stars in a sphere of radius } r) &= p(n=0) \\ &= \exp[-4/3\,\pi\rho r^3] \end{aligned}$$

Next consider the quantity p(at least one star in a shell between r and $r + dr$). When the differential volume element involved is so small that the expected number of stars within it is much less than one, this quantity can be replaced by p(exactly one star in a shell between r and $r + dr$). Now the volume element is that of the shell and $< n >= \rho\Delta V = 4\pi\rho r^2\,dr$. Thus

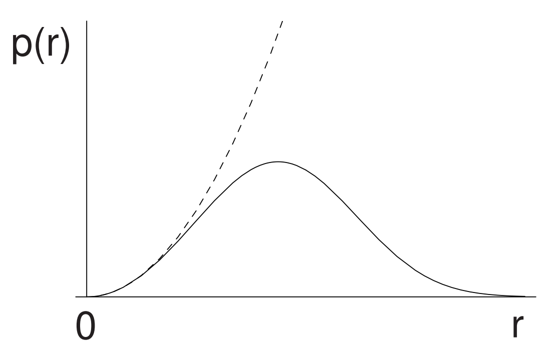
$$\begin{aligned} p(\text{at least one star in a shell between } r \text{ and } r+dr) &= p(n=1) \\ &= \underbrace{\frac{1}{1!}}_{=1} < n >^{(1)} \underbrace{e^{-<n>}}_{\approx 1} \\ &\approx\ < n >= 4\pi\rho r^2\,dr \end{aligned}$$

Now $p(r)$ is defined as the probability density for the event "the first star occurs between r and $r + dr$". Since the positions of the stars are (in this model) statistically independent, this can be written as the product of the two separate probabilities found above.

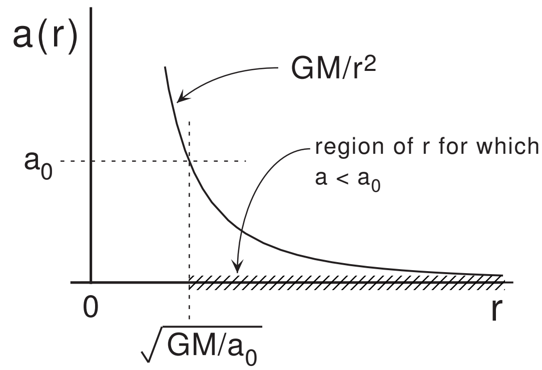
$$\begin{aligned} p(r)dr &= p(\text{no star out to } r) \times p(1 \text{ star between } r \text{ and } r+dr) \\ &= 4\pi\rho r^2 \exp[-4/3\,\pi\rho r^3]\,dr \end{aligned}$$

Dividing out the differential dr and being careful about the range of applicability leaves us with

$$\begin{aligned} p(r) &= 4\pi\rho r^2 \exp[-4/3\,\pi\rho r^3] \qquad r \geq 0 \\ &= 0 \qquad\qquad r < 0 \end{aligned}$$



b) Now we want to find the probability density function for the acceleration random variable a when $a = GM/r^2$, where r is the distance to a nearest neighbor star of mass M.



$$P(a) = \int_{\sqrt{GM/a}}^{\infty} p_r(r)\, dr$$

$$p(a) = \frac{dP(a)}{da} = \frac{1}{2a}\sqrt{\frac{GM}{a}}\; p_r\left(\sqrt{\frac{GM}{a}}\right)$$

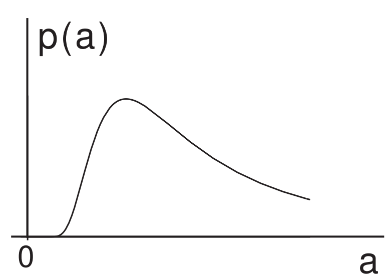
Distant neighbors will produce small forces and accelerations, so their effect on $p(a)$ will be greatest when a is small. [For a different approach, see the notes on the last page of this solution set.]

c) We use the $p_r(r)$ found in a):

$$p_r(r) = 4\pi\rho\, r^2\, e^{-4\pi\rho\, r^3/3}$$

to calculate

$$p(a) = \frac{2\pi\rho}{GM}\left(\frac{GM}{a}\right)^{5/2} e^{-(4\pi\rho/3)(GM/a)^{3/2}}$$



If there are binary stars and other complex units at close distances, these will have the greatest effect when r is small or when a is large.

d) The model considered here assumes all neighbor stars have the same mass M. To improve the model, one should consider a distribution of M values. One could even include the binary stars and other complex units, from part (b), with a suitable distribution.

Problem 4: Kinetic Energies in an Ideal Gasses

In this problem the kinetic energy E is a function of three random variables $(v_x,\, v_y,\, v_z)$ in three dimensions, two random variables $(v_x,\, v_y)$ in two dimensions, and just one random variable, v_x, in one dimension.

a) In three dimensions

$$E = \tfrac{1}{2}(v_x^2 + v_y^2 + v_z^2)$$

and the probability density for the three velocity random variables is

$$p(v_x,\, v_y,\, v_z) = \frac{1}{(2\pi\sigma^2)^{3/2}}\, e^{-(v_x^2+v_y^2+v_z^2)/2\sigma^2}.$$

Following our general procedure, we want first to calculate the cumulative probability function $P(E)$, which is the probability that the kinetic energy will have a value which is less than or equal to E. To find $P(E)$ we need to integrate $p(v_x,\, v_y,\, v_z)$ over all regions of $(v_x,\, v_y,\, v_z)$ that correspond to $\frac{1}{2}(v_x^2 + v_y^2 + v_z^2) \leq E$. That region in $(v_x,\, v_y,\, v_z)$ space is just a sphere of radius $\sqrt{2E/m}$ centered at the origin.

The integral is most easily done in spherical polar coordinates where

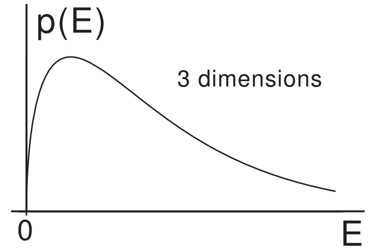
$$\begin{aligned}
v^2 &= v_x^2 + v_y^2 + v_z^2 \\
v_x &= v\, \sin\theta\, \cos\phi \\
v_y &= v\, \sin\theta\, \sin\phi \\
v_z &= v\, \cos\theta \\
dv_x\, dv_y\, dv_z &= v^2\, dv\, \sin\theta\, d\theta\, d\phi
\end{aligned}$$

This gives

$$\begin{aligned}
P(E) &= \frac{1}{(2\pi\sigma^2)^{3/2}} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi \int_0^{\sqrt{2E/m}} v^2\, e^{-v^2/2\sigma^2} dv \\
&= \frac{4\pi}{(2\pi\sigma^2)^{3/2}} \int_0^{\sqrt{2E/m}} v^2 e^{-v^2/2\sigma^2} dv
\end{aligned}$$

Thus (substituting $m\sigma^2 = kT$)

$$p(E) = \frac{dP(E)}{dE} = \frac{2}{\sqrt{\pi}} \frac{1}{kT} \sqrt{\frac{E}{kT}}\, e^{-E/kT} \quad \text{if } E \geq 0 \quad (0 \text{ if } E < 0)$$



You can use this to calculate $< E >= \frac{3}{2}kT$. [For a different approach, see the notes on the last page of this solution set.]

b) In two dimensions

$$E = \tfrac{1}{2}(v_x^2 + v_y^2)$$

and the probability density for the two velocity random variables is

$$p(v_x,\, v_y) = \frac{1}{(2\pi\sigma^2)}\, e^{-(v_x^2+v_y^2)/2\sigma^2}.$$

To find $P(E)$ we need to integrate $p(v_x,\, v_y)$ over all regions of $(v_x,\, v_y)$ that correspond to $\frac{1}{2}(v_x^2 + v_y^2) \leq E$. That region in $(v_x,\, v_y)$ space is just a circle of radius $\sqrt{2E/m}$ centered at the origin.

The integral is most easily done in polar coordinates where

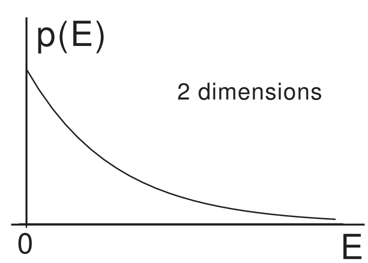
$$\begin{aligned} v^2 &= v_x^2 + v_y^2 \\ v_x &= v\,\cos\theta \\ v_y &= v\,\sin\theta \\ dv_x\,dv_y &= v\,dv\,d\theta \end{aligned}$$

This gives

$$P(E) = \frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\theta \int_0^{\sqrt{2E/m}} v\,e^{-v^2/2\sigma^2} dv = \frac{1}{\sigma^2}\int_0^{\sqrt{2E/m}} e^{-v^2/2\sigma^2} dv$$

Thus (substituting $m\sigma^2 = kT$)

$$p(E) = \frac{dP(E)}{dE} = \frac{1}{kT}\,e^{-E/kt} \;\text{ if } E \geq 0 \;\text{ (and 0 if } E < 0)$$



You can use this to calculate $< E >= kT$.

c) In one dimension (not required)

$$E = \tfrac{1}{2} v_x^2$$

and the probability density for the single velocity random variable is

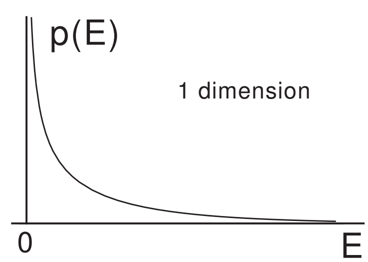
$$p(v_x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-v_x^2/2\sigma^2}.$$

To find $P(E)$ we need to integrate $p(v_x)$ over all regions of v_x that correspond to $\frac{1}{2}v_x^2 \leq E$. That region in v_x space is just a line from $-\sqrt{2E/m}$ to $\sqrt{2E/m}$.

$$P(E) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\sqrt{2E/m}}^{\sqrt{2E/m}} v\, e^{-\frac{v^2}{2\sigma^2}} dv$$

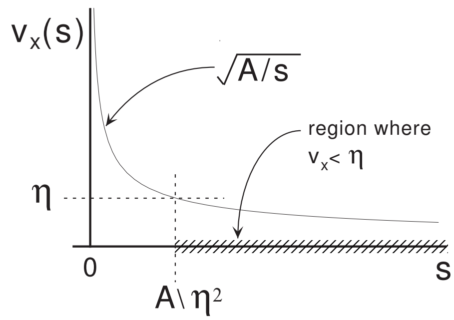
Thus (substituting $m\sigma^2 = kT$)

$$p(E) = \frac{dP(E)}{dE} = \frac{1}{\sqrt{\pi kT\, E}}\, e^{-E/kt} \quad \text{if } E \geq 0 \quad (0 \text{ if } E < 0)$$



You can use this to calculate $< E >= \frac{1}{2}kT$.

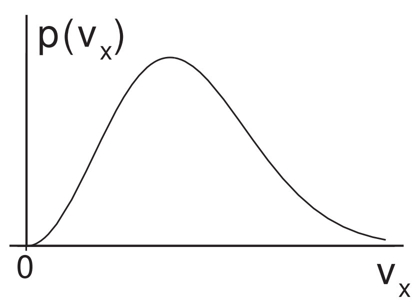
Problem 5: Atomic Velocity Profile



In this problem we have two random variables related by $s = A/v_x^2$. We are given $p_s(\zeta)$ and are to find $p(v_x)$. To find the probability $P(v)$ that $v_x \leq v$ we must calculate the probability that $s \geq A/v^2$. This is (the only possible values of v_x are ≥ 0)

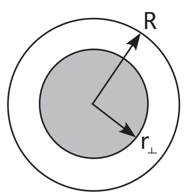
$$P(v_x) = \frac{A^{3/2}}{\sigma^2\sqrt{2\pi\sigma^2}} \int_{A/v^2}^{\infty} \frac{1}{\zeta^{5/2}} e^{-A/(2\sigma^2\zeta)} \, d\zeta$$

$$p(v_x) = \frac{dP(v_x)}{dv_x} = -\frac{A^{3/2}}{\sigma^2\sqrt{2\pi\sigma^2}} \left(\frac{v_x^2}{A}\right)^{5/2} e^{-v_x^2/2\sigma^2} \frac{d}{dv_x}\frac{A}{v_x^2} = \underline{\frac{2v_x^2}{\sigma^2\sqrt{2\pi\sigma^2}} e^{-v_x^2/2\sigma^2}}$$

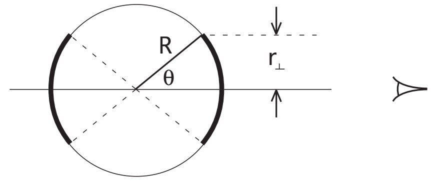


Problem 6: Planetary Nebulae

In this problem we are looking at matter distributed with equal probability over a spherical shell of radius R. When we look at it from a distance it appears as a ring because we are looking edgewise through the shell (and see much more matter) near the outer edge of the shell. If we choose coordinates so that the z axis points from the shell to us as observers on the earth, then $r_\perp = \sqrt{x^2 + y^2} = R\sin\theta$ will be the radius of the ring that we observe. The task for this problem is to find a probability distribution function $p(r_\perp)$ for the amount of matter that seems to us to be in a ring of radius $r_\perp$. This is a function of the two random variables θ and ϕ, which are the angular coordinates locating a point on the shell of the planetary nebula. Since the nebulae are spherically symmetric shells, we know that $p(\theta,\, \phi) = (4\pi)^{-1}$.



line of sight view



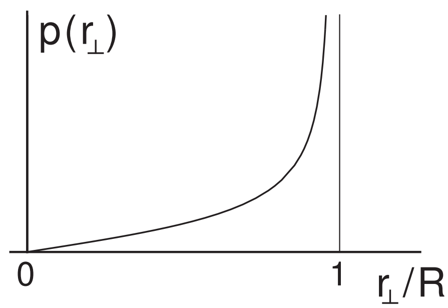
section view through center

We want first to find $P(r_\perp)$, the probability that the perpendicular distance from the line of sight is less than or equal to $r_\perp$. Since $r_\perp = R\sin\theta$, that is equal to the probability that $\sin\theta \le r_\perp/R$ as indicated in the figure above. This corresponds to the two polar patches on the nebula: $0 \le \theta \le \sin^{-1}(r_\perp/R)$ in the "top" hemisphere and $\sin^{-1}(r_\perp/R) \ge \theta \ge \pi$ in the "bottom" hemisphere.

$$P(r_\perp) = \frac{1}{4\pi}\int_0^{2\pi} d\phi \left(\int_0^{\sin^{-1}(r_\perp/R)} \sin\theta\, d\theta + \int_{\sin^{-1}(r_\perp/R)}^{\pi} \sin\theta\, d\theta \right) = 1 - \sqrt{1 - r_\perp^2/R^2}$$

$$p(r_\perp) = \frac{dP(r_\perp)}{dr_\perp} = \frac{r_\perp}{R\sqrt{R^2 - r_\perp^2}} \quad \text{for } 0 \le r_\perp \le R \text{ (and 0 otherwise)}$$

Note: $\cos\left(\sin^{-1}(r_\perp/R)\right) = \sqrt{1 - r_\perp^2/R^2}$.



[Insert for 3b]

Here, we have found the probability density with respect to the variable $a(r) = \frac{GM}{r^2}$ by first considering the cumulative probability,

$$P(a) = \int_{R=\left\{r:\frac{GM}{r^2}\leq a\right\}} p_r(r)\,dr,$$

where the integration region is over all values of r satisfying the inequality, and then taking taking the derivative to get the density:

$$p_a(a) \equiv \frac{dP(a)}{da} = \int_{\partial R=\left\{r:\frac{GM}{r^2}=a\right\}} p(r)\,dr = \int_{-\infty}^{\infty} p(r)\delta\left(\frac{GM}{r^2} - a\right)\,dr.$$

If one is comfortable with considering densities with respect to different variables, it is quicker to relate the densities directly using this last formula: the delta function just picks out the density with respect to a whenever $\frac{GM}{r^2} = a$. Here, the relation is monotonic, and so relating the densities is straight-forward, as the delta just picks out a single value. Recalling $\delta(f(x)) = \sum_{x:\,f(x)=0} \frac{\delta(x)}{|f'(x)|}$, we have

$$\begin{aligned}
p_a(a) &= \int_{-\infty}^{\infty} p_r(r)\delta\left(\frac{GM}{r^2} - a\right)\,dr \\
&= \int_{-\infty}^{\infty} p(r)\frac{\delta\left(r - \sqrt{\frac{GM}{a}}\right)}{\frac{da(r)}{dr}}\,dr \\
&= p_r(r)\left.\frac{dr}{da}\right|_{r=\sqrt{\frac{GM}{a}}} \\
&= \frac{2\pi\rho}{GM}\left(\frac{GM}{a}\right)^{\frac{5}{2}} e^{-(4\pi\rho/3)(GM/a)^{3/2}}
\end{aligned}$$

The two densities are just related by the Jacobian, as we would expect. The method can be used more generally, when the relationships between variables can be one to many, and when we consider probability distributions with more than one random variable.

[Insert for 4a]

Again, using the formula derived earlier relating the densities with respect to different variables, we can directly write

$$p_E(E) = \int_{-\infty}^{\infty} p_v(v)\,\delta\left(\frac{1}{2}mv^2 - E\right)\,dv.$$

The delta function now picks out the two solutions of the equality, and we find immediately

$$\begin{aligned}
p_E(E) &= \int_{-\infty}^{\infty} p_v(v)\,\frac{1}{m\,|v|}\left(\delta\left(v - \sqrt{\frac{2E}{m}}\right) + \delta\left(v + \sqrt{\frac{2E}{m}}\right)\right)\,dv \\
&= \left.\frac{p_v(v)}{mv}\right|_{v=+\sqrt{\frac{2E}{m}}} - \left.\frac{p_v(v)}{mv}\right|_{v=-\sqrt{\frac{2E}{m}}} \\
&= \frac{1}{\sqrt{\pi kT\,E}}e^{-\frac{E}{kT}}.
\end{aligned}$$

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