PROFESSOR: The fact is that angular momentum is an observable, and as such it deserves attention. There is an active way of thinking of observables, and we have not developed it that much in this course. But for example, with a momentum operator you've learned that the momentum operator can give you the differential operator. It's a derivative, and derivatives tell you how to move, how a function varies.

So with the momentum operator, for example, you have the momentum operator p hat, which is h bar over i d dx. And you could ask the question of, OK, so the momentum operator moves or takes a derivative, does the momentum operator move a function? Does it generate a translation? And the answer is, yes. That's another way of thinking of the momentum operator as a generator of translations.

But how does it do it? This is a Hermitian operator, and it takes a derivative. It doesn't translate the function. But there is a universal trick that if you exponentiate i times a Hermitian operator, you get a new kind of operator that actually, in this case, moves things. So we could think of exponentiating e to the i p hat, and for purposes of units I have to put a constant with units of length, and an h bar here. And now you have the exponential of an operator.

That's good. That's a very interesting operator, and we can ask what does it do when you act on a wave function? It's an operator. And look, simplify by putting what p is going to do. P is h over i d dx. So this is like a d dx exponentiated acting on psi of x. And as an exponential, it can be expanded in a Taylor series with this funny object there, but it would be the sum from n equals 0 to infinity 1 over n factorial a d dx. I will write it this as normal derivatives, because we just have a function of x, a d dx to the n psi of x.

And you see that, of course, this is psi of x plus a d psi dx plus 1 over 2 a squared, d second psi dx squared. But this is nothing else but the Taylor series for this. And there it is, the miracle. The e to the i momentum generated translation. It really moves the wave function. So that in a sense is a deeper way of characterizing the momentum operator as a generator of translations.

With the angular momentum operators, we will have that they generate rotations. So I need a little bit more mathematics here, because I have to deal with three dimensions, a vector, and produce an exponential that rotates the vector, so that it gives you the wave function at a rotated point. But this will be the same story. Angular momentum will generate rotations the same way as momentum generates translations.

And there is yet another story that when you will appreciate the abstract properties of angular momentum that some of them will appear today, you will realize that in addition of angular momentum that represent rotations of objects doing things, there is another way of having angular momentum. And that's spin angular momentum. That mysterious property of particles that have-- even though they have 0 size, they behave as if they were little balls rotating and spinning.

That spin angular momentum has no ordinary wave functions associated to it, and it's fractional sometimes. And the study of angular momentum inspired by orbital angular momentum associated with normal rotations, will lead us to understand where spin angular momentum comes about. So it's a gigantic interesting subject, and we're beginning with it today.

So it's really quantum mechanics in three dimensions, central potentials, and angular momentum. And let's begin by mentioning that if we are in three dimensions-- and many things with it so far in this course, we always took the time to write them in three dimensions. So we wrote this, for example, as a generalization of the derivative form of the momentum operator. Meaning there is a Px, which is h bar over i d dx, Py h bar over i d dy, and Pz equal h bar over i d dz.

And we had commutators within Px and x, Py and y, and Pz and z. There were always the same commutators of the form x Px equal i h bar. Similar things here. With this we wrote the three dimensional Schrodinger equation, which was minus h squared over 2m, and instead of p squared three dimensional, he would have a derivative if you were doing in one dimension. For three dimensions you have the Laplacian. And this time you have a wave function that depends on the vector x plus v of r-- v of x. Should I write r? Let me write r vector. V of r psi of r equal e psi of r.

This is our time independent Schrodinger equation. This corresponds to the energy eigenstate, but in three dimensions. So this is the equation we wish to understand, and our ability to understand that equation in a simple and nice way rests on a simplification. That is not always true, but it's true under so many circumstances that it's worth studying by itself. And it's the case when you have a central potential, and by that we mean that the potential is not quite the vector function of r, but is just a function of the magnitude of r.

That's a little bit funny way of writing it, because I'm using the same letter v, but I hope there's no confusion. I mean that the potential just depends on the value of r. So what this means physically is that over concentric spheres, the potential is constant. All over the surface of spheres of constant radius, the potential is constant, because it only depends on the radius.

And this potential is there for a spherically symmetric. You can rotate the world, and the potential still looks the same, because rotations don't change the magnitude of vectors. If you have a vector of some length, you rotate it, it's the same length, and therefore you remain on this sphere. So the central potential are spherically symmetric. By that we mean they're invariant under rotations.

So this is the reason why angular momentum will play an important role, because precisely the angular momentum operators, in the fashion we discussed a minute ago, generate rotation. So they will have a nice interplay, to be developed in the following lectures, with the Hamiltonian. So at this moment we have a central potential, and let's assume that's the case. And we need to understand a little more of this differential equation. So let me write the formula for the Laplacian of a function.

It has a radial contribution. You know it's second order derivatives. And it has a radial part, and an angular part. The units are 1 over length squared. So you need, if you have an angular part, all over here is going to be angular, you still need the 1 over on r squared here for the units to work out.

So here it is, it's slightly complicated. d d theta sine theta d d theta of-- well, put the psi, plus 1 over sine squared theta, d second d phi squared all acting on psi. It's a complicated operator, and here is some radial derivatives, and here there are some angular derivatives. So you see, today's lecture will have many steps, and you have to keep track of where we're going. And what we're going to do is, build up a structure that allows us pretty much to forget about all this thing. That's our goal.

And angular momentum will play a role in doing this. So there are in fact two things I want to justify, two facts to be justified. So I will erase this. The first fact is the relation between this differential operator and angular momentum. So two facts to justify. The first is that minus h squared 1 over sine theta d d theta sine theta d d theta plus 1 over sine squared theta d second d phi squared.

This whole thing can be viewed as the differential operator version of angular momentum. Remember, d dx was a differential operator version of momentum. So maybe this has to do with angular momentum, and indeed this whole thing, remember, units of angular momentum is h bar. Angular momentum is length times momentum. And from the certainty principle, you know that x times p has units of h bar. So angular momentum has units of h bar. So there's h bar squared here. So this must be angular momentum squared.

In fact, if you think about that, angular momentum is x times p. So x times a derivative. So it's a first order differential operator, but this is a second order one. So this could not be just angular momentum. Anyway, angular momentum is a vector. So this will turn out to be, and we will want to justify L squared. The quantum version of the angular momentum operator squared.

And the other thing I want to justify if I write-- call this equation one. So this is fact one, and fact two, is that equation one is relevant, when-- let me wait a second to complete this. This equation is an equation for a particle moving in a potential, a spherically symmetric potential. It turns out that is relevant under more general circumstances. If you have two particles whose potential energy-- if you have two particles you have a potential energy between them, maybe it's a electromagnetic-- if the potential energy just depends on the distance that separates them, this two body problem can be reduced to a one body problem of this form.

This is a fairly non-trivial fact, and an absolutely interesting one. Because if you want to really solve the hydrogen atom, you have an electron and a proton. Now it turns out that the proton is almost 2,000 times heavier than the electron. And therefore, you could almost think that the proton creates a potential in which the electron moves.

But similar analysis is valid for neuron orbiting a nucleus. And in that case, the neuron is still lighter than the proton, but not that much lighter. Or maybe for a quark and an anti-quark orbiting each other. Or an electron and a positron orbiting each other, and this would be valid and useful. So we need to somehow explain that as well.

If you really want to understand what's going on, is that equation one is relevant when we have a two body problem with a potential function v of x1 x2. The potential energy given that configuration, x1 and x2 of the first and second particle, is a function of the separation only. The absolute value or the length of the vector, it's 1 minus x2.

This far we'll get through today. This will be next lecture still.