PROFESSOR: Simultaneous eigenstates. So let's begin with that. We decided that we could pick 1 l and l squared, and they would commute. And we could try to find functions that are eigenstates of both.

So if we have functions that are eigenstates of those, we'll try to expand in terms of those functions. And all this operator will become a number acting on those functions. And that's why the Laplacian simplifies, and that's why we'll be able to reduce the Schrodinger equation to a radial equation. This is the goal.

Schrodinger equation has r theta and phi. But theta and phi will deal with all the angular dependents. We'll find functions for which that operator gives a number acting on them. And therefore, the whole differential equation will simplify.

So simultaneous eigenstates, and given the simplicity of l z, everybody chooses l z. So we should find simultaneous eigenstates of this two things. And let's call them psi l m of theta and phi. Where l and m are numbers that, at this moment, are totally arbitrary, but are related to the eigenvalues of this equations.

So we wish that l z acting on psi l m is going to be a number times psi l m. That is to be an eigenstate. The number must have the right units, must be an H bar. And then we'll use m. We don't say what m is yet. M. Where m belongs to the real numbers. Because the eigenvalues of a Hermitian operator are always real.

So this could be what we would demand from l z. From l squared on psi l m, I can demand that this be equal because of units and h squared. And then a number, lambda psi l m. Now this lambda-- do I know anything about this lambda?

Well, I could argue that this lambda has to be positive. And the reason is that this begins as some sort of positive operator, is L. Squared. Now that intuition may not be completely precise. But if you followed it a little more with an inner product. Suppose we would have an inner product, and we can put psi l m here. And l squared, psi l m from this equation. This would be equal to h squared lambda, psi l m, psi l m. An inner product if you have it there.

And then if your wave functions are suitably normalized, this would be a 1. But this thing is l x-- l x plus l y, l y plus l z, l z. And l x, l x-- you could bring one l x here, and you would have l x, psi l m, l x psi l m. Plus the same thing for y and for z.

And each of these things is positive. Because when you have the same wavefunction on the left and on the right, you integrate the norm squared. It's positive. This is positive. This is positive. So the sum must be positive, and lambda must be positive.

So lambda must be positive. This is our expectation. And it's a reasonable expectation. And that's why, in fact, anticipating a little the answer, people write this as l times l plus 1 psi l m. And where l is a real number, at this moment. And you say, well, that's a little strange. Why do you put it as l times l plus 1. What's the reason?

The reason is-- comes when we look at the differential equation. But the reason you don't get in trouble by doing this is that as you span all the real numbers, the function l times l plus 1 is like this. l times l plus 1. And therefore, whatever lambda you have that is positive, there is some l for which l times l plus 1 is a positive number.

So there's nothing wrong. I'm trying to argue there's nothing wrong with writing that the eigenvalue is of the form l times l plus 1. Because we know the eigenvalue's positive, and therefore, whatever lambda you give me that is positive, I can always find, in fact, two values of l, for which l times l plus 1 is equal to lambda. We can choose the positive one, and that's what we will do.

So these are the equations we want to deal with. Are there questions in the setting up of these equations? This is the conceptual part. Now begins a little bit of play with the differential equations. And we'll have to do a little bit of work. But this is what the physical intuition-- the commutators, everything led us to believe. That we should be able to solve this much. We should be able to find functions that do all this.

All right, let's do the first one. So the first equation-- The first equation is-- let me call it equation 1 and 2. The first equation is h bar over i d d 5. That's l z, psi l m, equal h bar m psi l m. So canceling the h bars, you'll get dd phi of psi l m is equal to i m, psi l m.

So psi l m is equal to e to the i m phi times some function of theta. Arbitrary function of theta this moment. So this is my solution. This is up psi l m of theta and phi. With the term in the phi dependants, and it's not that complicated.

So at this moment, you say, well, I'm going to use this for wavefunctions. I want them to behave normally. So if somebody gives me a value of phi, I can tell them what the wavefunction is. And since phi increases by 2 pi and is periodic with 2 pi, I may demand that psi l m of theta, and 5 plus 2 pi be the same as psi l m theta and phi.

You could say, well, what if you could put the minus sign there? Well, you could try. The attempt would fail eventually. There's nothing, obviously, wrong with trying to put the sine there. But it doesn't work. It would lead to rather inconsistent things soon enough. So this condition here requires that this function be periodic.

And therefore when phi changes by 2 pi, it should be a multiple of 2 pi. So m belong to the integers. So we found the first quantization. The eigenvalues of l z are quantized. They have to be integers. That was easy enough. Let's look at the second equation. That takes a bit more work.

So what is the second equation? Well, it is most slightly complicated differential operator. And let's see what it does. So l squared. Well, we had it there. So it's minus h squared 1 over sine theta, dd theta, sine theta, dd theta, plus 1 over sine squared theta, d second d phi squared psi l m equal h squared l times l plus 1 psi l m.

One thing we can do here is let the dd phi squared act on this. Because we know what dd phi does. Dd phi brings an i n factor, because you know already the phi dependents of psi l m. So things we can do. So we'll do the second d 5 squared gives minus-- gives you i m squared, which is minus m squared, multiplying the same function.

You can cancel the h bar squared. Cancel h bar squared. And multiply by minus sine squared theta. To clean up things. So few things. So here is what we have. We have sine theta, dd theta. This is the minus sine squared that you are multiplying. The h squared went away. Sine theta, d p l m d theta.

Already I substituted that psi was into the i m phi times the p. So I have that. And maybe I should put the parentheses here to make it all look nicer. Then I have in here two more terms. I'll bring the right-hand side to the left. It will end up with l l plus 1, sine squared theta, minus m squared, p l m equals 0.

There we go. That's our differential equation. It's a major, somewhat complicated, differential equation. But it's a famous one, because it comes from [? Laplatians. ?] You know, people had to study this equation to do anything with Laplatians, and so many problems. So everything is known about this.

And the first thing that is known is that theta really appears as cosine theta everywhere. And that makes sense. You see, theta and cosine theta is sort of the same thing, even though it doesn't look like it. You need angles that go from 0 to pi. And that's nice.

But [? close ?] and theta, in that interval goes from 1 to minus 1. So it's a good parameter. People use 0 to 180 degrees of latitude. But you could use from 1 to minus 1, the cosine. That would be perfectly good. So theta or cosine theta is a different variable. And this equation is simpler for cosine theta as a variable. So let me write that, do that simplification.

So I have it here. If x is closer in theta, d d x is minus 1 over sine theta, d d theta. Please check that. And you can also show that sine theta, d d theta is equal to minus 1 minus x squared d d x.

The claim is that this differential equation just involves cosine theta. And this operator you see in the first term of the differential equation, sine theta, dd theta is this, where x is cosine theta. And then there is a sine squared theta, but sine squared theta is 1 minus cosine squared theta.

So this differential equation becomes d d x-- well, should I write the whole thing? No. I'll write the simplified version. It's not-- it's only one slight-- m of the x plus l times l plus 1 minus m squared over 1 minus x squared p l m of x equals 0. The only thing that you may wonder is what happened to the 1 minus x squared that arises from this first term.

Well, there's a 1 minus x squared here. And we divided by all of it. So it disappeared from the first term, disappeared from here. But the m squared ended up divided by 1 minus x squared. So this is our equation. And so far, our solutions are psi l m's. Are going to be some coefficients, m l m's, into the i m phi p l m of cosine theta.

Now I want to do a little more before finishing today's lecture. So this equation is somewhat complicated. So the way physicists analyze it is by considering first the case when m is equal to 0. And when m is equal to 0, the differential equation-- m equals 0 first. The differential equation becomes d d x 1 minus x squared d p l 0. But p l 0, people write as p l. The x plus l times l plus 1, p l equals 0.

So this we solve by a serious solution. So we write p l of x equals some sort of a k-- sum over k, a k, x k. And we substitute in there. Now if you substituted it and pick the coefficient of x to the k, you get a recursion relation, like we did for the case of the harmonic oscillator.

And this is a simple recursion relation. It reads k plus 1-- this is a two-line exercise-- k plus 2, a k plus 2, plus l times l plus 1, minus k times k plus 1, a k. So actually, this recursive relation can be put as a [? ratio ?] form. The [? ratio ?] form we're accustomed, in which we divide a k plus 2 by a k. And that gives you a k plus 2 over a k. I'm sorry, all this coefficient must be equal to 0.

And a k plus 2 over a k, therefore is minus l times l plus 1 minus k times k plus 1 over k plus 1 times k plus 2. OK, good. We're almost done. So what has happened? We had a general equation for phi. The first equation, one, we solved.

The second became an [? integrated ?] differential equation. We still don't know how to solve it. M must be an integer so far. L we have no idea. Nevertheless we now solve this for the case m equal to 0, and find this recursive relation. And this same story that happened for the harmonic oscillator happens here.

If this recursion doesn't terminate, you get singular functions that diverge at x equals 1 or minus 1. And therefore this must terminate. Must terminate. And if it terminates, the only way to achieve termination on this series is if l is an integer equal to k. So you can choose some case-- you choose l equals to k. And then you get that p l of x is of the form of an x to the l coefficient.

Because l is equal to k, and a k is the last one that exists. And now a l plus 2, k plus 2 would be equal to 0. So you match this, the last efficient is the value of l. And the polynomial is an elf polynomial, up to some number at the end. and you got a quantization. L now can be any positive integer or 0.

So l can be 0, 1, 2, 3, 4. And it's the quantization of the magnitude of the angular momentum. This is a little surprising. L squared is an operator that reflects the magnitude of the angular momentum. And suddenly, it is quantized. The eigenvalues of that operator, where l times l plus 1, that I had in some blackboard must be quantized.

So what you get here are the Legendre polynomials. The p l's of x that satisfy this differential equation are legendre polynomials. And next time, when we return to this equation, we'll find that m cannot exceed l. Otherwise you can't solve this equation. So we'll find the complete set of constraints on the eigenvalues of the operator.