BARTON ZWIEBACH: We want to understand now our observables. So we said these are observables, so can we observe them? Can we have a state in which we say, what is the value of Lx, the value of Ly, and the value of Lz. Well, a little caution is necessary because we have states and we have position and momentum operator and they didn't commute and we ended up that we could not tell simultaneously the position and the momentum of a state. So for this angular momentum operators, they don't commute, so a similar situation may be happening.

So I want to explain, for example, or ask, can we have simultaneous eigenstates of Lx, Ly, and Lz? And the answer is no. And let's see why that happens.

So let's assume we can have simultaneous eigenstates and let's assume, for example, that Lx on that eigenstate phi nought is some number lambda x phi nought, and Ly and phi nought is equal to lambda y phi nought. Well, the difficulty with this is essentially-- well, we could even say that Lz on phi nought is equal to lambda z phi nought. So what is the complication? The complication are those commutators. If you do Lx, Ly and phi nought, you're supposed to get i h-bar Lz and phi nought. And therefore, you're supposed to get i h-bar lambda z times phi nought, because it's supposed to be an eigenstate.

But how about the left hand side? The left hand side is LxLy and phi nought minus LyLx and phi nought. When Ly acts, it produces a lambda y, but then phi nought, and then when Lx acts, it produces a lambda x, so this produces lambda x lambda y phi nought minus lambda y lambda x phi nought, which is the same thing, so the left hand side is 0. 0 is equal to lambda z phi nought, so you get a-- lambda z must be 0. If you have a non-trivial state, lambda z should be 0.

By the other commutators-- this can be attained or applied to phi nought-- would be 0 again, because each term produces a number and the order doesn't matter. But then it would show that lambda x is 0, and this will show that lambda y is 0. So at the end of the day, if these three things hold, then all of them are 0. Lambda x equals lambda y equals lambda z equals 0. So you can have something that is killed by all of the operators, but you cannot have a non-trivial state with non-trivial eigenvalues of these things. So we cannot have-- we cannot tell what is Lx on this state and Ly on this state simultaneously. Any of those two is too much.

So if we can't tell that, what can we tell? So what is the most we can tell about this state Is our question now. We can tell maybe what is its value of Lx, but then Ly and Lz are undetermined. Or we can tell what is Lz and then Lx and Ly are undetermined, incalculable, impossible in principle to calculate them. So let's see what we can do, and the answer comes from a rather surprising thing, the fact that if you think about what could commute with Lx, Ly, Lz, it should be a rotationally invariant thing, because Lx, Ly, and Lz do rotations. So the only thing that could possibly commute with this thing is something that is rotationally invariant.

The thing that could work out is some thing that is invariant and there are rotations. Now we said, for example, the magnitude of the vector R is invariant under rotation. You rotate the vector, the demanded is invariant. So we can try the operator L squared, which is proportionate to the magnitude squared, so we define it to be LxLx plus LyLy plus LzLz. And we tried, we tried to see if maybe Lx commutes with L squared.

Remember, we had a role for L squared in this differential operator that had the Laplacian, the angular part of the Laplacian was our role for L squared, so L squared is starting to come back. So let's see here-- this is Lx-- now, I'll write the whole thing-- LxLx plus LyLy plus LzLz. Now, Lx and Lx commute, so I don't have to bother with this thing, that's 0. But the other ones don't commute. So let's do the distributive law. So this would be an Lx, Ly Ly plus Ly Lx, Ly-- this is from the first-- plus Lx, Lz Lz plus Lz Lx, Lz. You know, if you don't put these operators in the right order, you don't get the right answer. So I think I did. Yes. It's correct.

Now you use the commutators and hope for the best. So Lx, Ly is i h-bar LzLy. Lx, Ly is plus i h-bar LyLz. So far, no signs of canceling, these two things are very different from each other. They don't even appear with a minus sign, so this is not a commutator, but anyway, what is this? Lx with Lz. Well, you should always think cyclically. Lz with Lx is i h-bar, so this would be minus i h-bar LyLz, and this is again Lz with Lx would have been i h-bar Ly, so this is minus i h-bar LzLy, and it better cancel-- yes. This term cancels with the first and this term cancels with this and you get 0.

That's an incredible relief, because now you have a second operator that is measurable simultaneously. You can get eigenstates that are eigenstates of one of the L's-- for example, Lx and L squared, because they commute, and you won't have the problems you have there. In fact, it's a general theorem of linear algebra that-- we'll see a little bit of that in this course and you'll see it more completely in 805-- that if you have two Hermitian operators that commute, you can find a simultaneous eigenstates of both operators. I mean, eigenstates that are eigenstates of 1, and eigenstates of the second. Simultaneous eigenstates are possible.

So we can find simultaneous eigenstates of these operators, and in fact, you could find simultaneous eigenstates of Lx and L squared, but given the simplicity of all this, it also means that Ly commutes with L squared, and that Lz also commutes with L squared. So you have a choice-- you can choose Lx, Ly, or Lz and L squared and try to form simultaneous eigenstates from all these operators. Two of them. Let's study those operators as differential operators a little bit.

So x, y, and z are your spherical coordinates and they are r sin theta cos phi, r sin theta sin phi, and r cos theta. We're trying to calculate the differential operators associated with angular momentum using spherical coordinates. So r is x squared plus y squared plus z squared. Theta is cosine minus 1 of z over r and 5 is tan minus 1 y over z. And there's something very nice about one angular momentum operator in spherical coordinates, there is only one angular momentum that is very simple-- its rotations about z. Rotations about z don't change the angle theta of spherical coordinates, just change the angle phi. r doesn't change. The other rotation, the rotation about x messes up phi and theta and all the others are complicated, so maybe we can have some luck and understand what is d/d-phi, the d/d-phi operator.

Well, the d/d-phi operator is d/dy dy/d-phi plus d/dx dx/d-phi-- the rules of chain rule for partial derivatives-- plus d/dz dz/d-phi. But z doesn't depend on phi. On the other hand, dy/d-phi is what? dy/d-phi, this becomes a cos phi-- it's x. X d/dy. And dx/d-phi is minus y. And you say, wow, x d/dy is like x py minus y px, that's a z component of angular momentum!

So indeed, Lz, which is h-bar over i x d/dy minus y d/dx, h-bar over i is because of the p's-- x py minus y px. And this thing is d/d-phi, so Lz, we discover, is just h-bar over i d/d-phi. A very nice equation that tells you that the angular momentum in the z direction is associated with its operator.

So I have left us exercises to calculate the other operators that are more messy, and to calculate Lx Ly as well in terms of d/d-theta's and d/d-phi's. And as you remember, angular momentum has units of h-bar and angles have no units, so the units are good and we should find that. So that calculation is left as an exercise, but now you probably could believe that L squared, which is LxLx plus LyLy plus LzLz is really minus h squared 1 over sin theta d/d-theta. No, it's-- not 1 over sin theta-- uh, yep. 1 sin theta d/d-theta-- sin theta d/d-theta plus 1 over sin squared theta d second d phi squared.

So the claim that they had relating the angular momentum operator to the Laplacian is true. But, you know, you now see the beginning of how you calculate these things, but it will be a simple and nice exercise for you to do it.