PROFESSOR: We're talking about angular momentum. We've motivated angular momentum as a set of operators that provided observables, things we can measure. Therefore, they are important.

But they're particularly important for systems in which you have central potentials. Potentials that depend just on the magnitude of the radial variable. A v of r that depends just on the magnitude of the vector r relevant to cases where you have two bodies interacting through a potential that just depends on the distance between the particles.

So what did we develop? Well, we discussed the definition of the angular momentum operator. You saw they were permission. We found that they satisfy a series of commutators in which lx with ly gave ih bar lz, and cyclical versions of that equation, which ensure that actually you can measure simultaneously the three components of angular momentum. You can measure, in fact, just one. Happily we found there was another object we could measure, which was the square of the total angular momentum.

Now, you should understand this symbol. It's not a vector. It is just a single operator. l squared is, by definition, lx times lx, plus ly times ly, plus lz times lz. This is this operator.

And we showed that any component of angular momentum, be it lx, ly, or lz, commutes with l squared. Given that they commute, it's a general theorem that two permission operators that commute, you can find simultaneous eigenstates of those two operators. And therefore, we set up for the search of those wave functions that are simultaneous eigenstates of one of the three components of angular momentum. Everybody chooses lz and l squared. lz being proportional to angular momentum has an h bar m.

We figured out by looking at this differential equation that if we wanted single valued wave functions-- wave functions would be the same at phi, and at phi plus 2 pi, which is the same point. You must choose m to be an integer. For the l squared operator we also explained that the eigenvalue of this operator should be positive. That is achieved when l, whatever it is, is greater than 0, greater or equal than 0. And the discussion that led to the quantization of l was a little longer, took a bit more work.

Happily we have this operator, and operator we can diagonalize, or we can find eigenstates for it. Because the Laplacian, as was written in the previous lecture, Laplacian entering in the Schrodinger equation has a radial part and an angular part, where you have dd thetas, and sine thetas, and the second defies square. All these things were taken care of by l squared. And that's very useful.

Well, the differential equation for l squared-- this can be though as a differential equation-- ended up being of this form, which is of an equation for the so-called Associate Legendre functions. For the case of m equals 0 it simplifies very much so that it becomes an equation for what were eventually called Legenre polynomials.

We looked at that differential equation with m equals 0. We called it pl 0. So we don't write the zeros. Everybody writes pl for those polynomials.

And looking at the differential equation one finds that they have divergences at theta equals 0, and a theta equal pi, north and south pole of this spherical coordinate system. There aren't divergences unless these differential equations has a polynomial solution that this is serious the recursion relations terminate.

And that gave for us the quantization of l. And that's where we stopped. These are the Legendre polynomials. Solve this equation for m equals 0.

Are there any questions? Anything about the definitions or? Yes?

AUDIENCE: Why do we care about simultaneous eigenstates?

PROFESSOR: Well, the question is why do we care about simultaneous eigenstates. The answer is that if you have a system you want to figure out what are the properties of the states. And you could begin by saying the only thing I can know about this state is its energy. OK, well, I know the energy at least.

But maybe thinking harder you can figure out, oh, you can also know the momentum. That's progress. If you can also know the angular momentum you learn more about the physics of this state. So in general, you will be led in any physical problem to look for the maximal set of commuting operators. The most number of operators that you could possibly measure. You know you have success at the very least, if you can uniquely characterize that states of the system by observables.

Let's assume you have a particle in a circle. Remember that the free particle in a circle has degenerate energy eigenstates. So you have two energy eigenstates for every allowed energy, except for 0 energy, but two energy eigenstates. And you would be baffled.

You'd say, why do I have two? There must be some difference between these two states. If there are two states, there must be some property that distinguishes them. If there is no property that distinguishes them, they should be the same state. So you're left to search for another thing.

And in that case the answer was simple. It was the momentum. You have a particle with some momentum in one direction, or in the reverse direction. So in general, it's a most important question to try to enlarge the set of commuting observables. Leading finally to what is initially called a complete set of commuting observables.

So what do we have to do today? We want to complete this analysis. We'll work back to this equation. And then work back to the Schrodinger equation to finally obtain the relevant differential equation we have to solve if you have a spherical symmetric potential. So the equation will be there in a little while.

Then we'll look at the hydrogen atom. We'll begin the hydrogen atom and this task why? Having a proton and an electron we can reduce this system to as if we had one particle in a central potential. So that will be also very important physically.

So let's move ahead. And here there is a simple observation that one can make. Is that the differential equation for p l m depends on m squared.

We expect to need values of m that are positive and negative. You have wave functions here, of this form. The complex conjugate ones should be thought as having m negative. So we expect positive and negative m's to be allowed.

So how did people figure this out? They, in fact, figured out that if you have these polynomials you can create automatically the solutions for this equation. There's a rule, a simple rule that leads to solutions. You put p l m of x is equal to 1 minus x squared, to the absolute value of m over 2.

So there are square roots here, possibly. An absolute value of m means that this is always in the numerator, whether m is positive or negative. And d, dx acting exactly absolute value of m times on p l x. The fact is that this definition solves the differential equation star.

This takes a little work to check. I will not check it, nor the notes will check it. It's probably something you can find the calculation in some books. But it's not all that important. The important thing to note here is the following.

That this provides solutions. Since this polynomial is like x to the l plus x to the l minus 2 plus coefficients like this. You can think that most m equal l derivatives-- if you take more than l derivatives you get 0. And there's no great honor in finding zero solution of this equation. These are no solutions.

So this produces solutions for an, absolute value of m, less than l. So produces solutions for absolute value of m less or equal to l. And therefore m in between l and minus l.

But that's not all that happens. There's a little more that takes mathematicians some skill to do. It's to show that there are no more solutions. You might seem that you were very clever and you found some solutions, but it's a theorem that there are no more solutions. No additional regular solutions. I mean solutions that don't diverge.

So this is very important. It shows that there is one more constraint on your quantum numbers. This formula you may forget, but you should never forget this one. This one says that if you choose some l which corresponds to choosing the magnitude of the angular momentum, l is the eigenvalue that tells you about the magnitude of the angular momentum.

You will have several possibilities for m. There will be several states that have the same l, but m different.

So for example you'll have l equal 0, in which case m must be equal to 0. But if you choose state with l equals 1, or eigenfunctions with l equal 1, there is the possibility of having m equals minus 1, 0, or 1. So are three waves functions in that case. Psi 1, minus 1, psi 1, 0, and psi 1, 1.

So in general when we choose a general l, if you choose an arbitrary l, then m goes from minus l, minus l plus 1 all the way up to l. These are all the values which are 2l plus 1 values. 2l and the 0 value in between. So it's 2l plus 1 values.

The quantization in some sense is done now. And let me recap about these functions now. We mentioned up there that the y l m's are the objects. The spherical harmonicas are going to be those wave functions. And they have a normalization, n l m, an exponention, and all that.

So let me write, just for the record, what a y l m looks like with all the constants. Well, the normalization constant is complicated. And it's kind of a thing you can never remember by heart. It would be pointless. OK. All of that.

Then a minus 1 to the m seems useful. e to the i m phi, p l m of cosine theta. And this is all valid for 0 less 0 m positive m. When you have negative m you must do a little variation for m less than 0 y l m of theta and phi is minus 1 to the m y l minus m of theta and phi complex conjugated.

Well, if m is negative, minus m is positive. So you know what that is. So you could plug this whole mess here. I don't advise it. It's just for the record.

These polynomials are complicated, but they are normalized nicely. And we just need to understand what it means to be normalized nicely. That is important for us. The specific forms of these polynomials we can find them.

The only one I really remember is that y 0 0 is a constant. It's 1 over 4 pi. That's simple enough. No dependents. l equals 0, m equals 0.

Here is another one. y1 plus minus 1 is minus plus square root of 3 over 8 pi e to the plus minus i phi sine theta. And the last one, so we're giving all the spherical harmonics with l equals 1. So with l equals 1 remember we mentioned that you would have three values of m. Here they are. Plus or minus 1 and 0.