PROFESSOR: So angular momentum, we need to deal with angular momentum, and the inspiration for it is classical. We have L is r cross p. Classically. So let's try to just use that information and write the various operators. And in fact, we're lucky in this case. The operators that we would write inspired by the classical definition are good operators and will do the job.

So what do we have? Lx, if you remember the cross-product rule, that would be y Pz minus z Py. Now you can think of this thing as a cyclic, like a circle when you have x and Px, y and Py, and z and Pz. Things are cyclically symmetric. There's no real difference between this core. And so you can go cyclically here.

So you say, let's go cyclical on this index. Ly is equal to the next cyclic to y in that direction is z Px minus x Pz. And Lz is equal to x Py minus y Px. And these things, I'll think of them as the operators. Let's put hats to everything.

The first thing I can wonder with a little bit of trepidation is maybe I got the ordering wrong. Should I have written-- here classically, you put r cross p, and then the order of these two terms doesn't matter. Does it matter quantum mechanically? Happily, it doesn't matter because y and Pz commute.

z and Py commute, so you could even have written them the other way, and they are good. All of them are ambiguous. You could have even written them the other way, and they would be fine.

But now these are operators. And moreover, they are Hermitian operators. Hermitian. Let's see. Lx dagger. Well, the dagger of two operators you would do Pz dagger y hat dagger-- recall the dagger changes the order-- minus Py dagger z dagger.

Now, p and x's are all for Hermitian operators, so this is Pz y minus Py z. And we use, again, that y and Pz commute, and z and Py commute to put it back in the standard form. And that's, again, Lx. So it is an Hermitian operator.

And so is Ly and Lz. That means these operators are observables. That's all you need for the operator to be an observable. And that's a very good thing. So these operators are observables. Li's are observable. But they're funny properties. With these operators, they're not all that simple in some ways.

So next we have these operators. Whenever you have quantum operators, the thing you do next is compute their commutators Just like we did with x and p, we wanted to know what that commutator is. We want to know what is the commutator of this L operator. So we'll do Lx with Ly. Try to compute the commutators.

So Lx is y Pz. Let me forget the hat, so basically minus z Py. And Ly is z Px minus x Pz. Here is a y. The y commutes with everything here, so the y doesn't get. The Pz gets stuck with the z and doesn't care about this. So this term just talks to that term.

And here the z Py, the Py doesn't care about anybody here, but this z, well, doesn't care about that z, but it does care about this Pz. So the only contribution, there could have been four terms out of this commutator, but only two are relevant.

So let's write them down. y Pz with z Px and minus, it's a plus z Py x Pz. Well, you can start peeling off things. You can think of this as a single operator with this too, and it will fail to commit with the first. So you have y Pz z Px. That's all this commutator gets.

And the same thing here. This fails to commute just with Pz, so the x can go out, x z Py Pz. And then here the y actually can go out, doesn't care about this z, goes out on the left. Not that it matters much here, but that's how using the commutator identities does. And this Py can go out and let's go out on the right, z Pz. And basing this on this identity, we just have A BC commutator and then AB C commutators, how things distribute.

Now, this is minus i h bar, and this is i h bar. So here we get i h bar x Py minus y Px. See everything came out in the right position. And you recognize that operator as Lz.

So this commutator here has given you Lx with Ly equal i h bar Lz. It's a very interesting and fascinating property that somehow you're doing this commutator, it could have been a mess, but it combined to give you another angular momentum operator.

Now, it looks like a miracle, but physically, it's not that miraculous. It actually has to do with the concept of symmetry. Symmetry transformations. If you have a symmetry transformation and you do commutators within those symmetry operators, you must get an operator that corresponds to that symmetry, or you must get a symmetry at the very least.

So if we say that the potential has very close symmetry, that suggests that when you do operations with these operators that generate rotation, you should get some rotation here.

And alternatively, although, again, this is suggestive, it can be made very precise, when you do rotations in different order, you don't get the same thing at the end.

Everybody knows if you have a page and you do one rotation and then the other as opposed to the other and then the first one, you don't get the same thing. Rotations do not commute. A single rotation does commute in one direction, but rotations in different directions don't commute. That is the reason for this equation.

And this equation, as we said, everything is cyclic. so you don't have to work again to argue that then Ly Lz, going cyclic, must be equal to i h bar Lx. And that Lz Lx must be i h bar Ly. And this is called the quantum algebra of angular momentum.

In fact, it is so important that this algebra appears in all fields of physics and mathematics, and all kinds of things show up. This algebra is related to the algebra of generators of the group SU2, Special Unitary Transformations in Two Dimensions.

It is related to the orthogonal group in three dimensions where you rotate things in three-dimensional space. It is here, the algebra of operators and in a sense, it's a deeper result than the derivation. It is one of those cases when you start with something very concrete and you suddenly discover a structure that is rather universal. Because we started with very concrete representation of L's in terms of y P's and all these things. But then they form a consistent unit by themselves.

So sometimes there will be operators that satisfy these relations, and they don't come from x's and P's, but still they satisfy that. And that's what happens with spin angular momentum. The spin angular momentum operators will be denoted with Sx, for example, and Sy will have i h bar spin in the z direction, and the others will follow.

But nevertheless nobody will ever be able to write spin as something like that because it's not, but spin exists. And it's because this structure is more general than the situation that allowed us to discover it. It's a lot more general and a lot more profound.

So in fact, mathematicians don't even mention angular momentum. They say, let's study. The subject of Lie algebra is the subject of classifying all possible consistent commutation relations. And this is the first non-trivial example they have, and they studied the books on this algebra.