PROFESSOR: We have to ask what happens here? This series for h of u doesn't seem to stop. You go a 0, a 2, a 4. Well, it could go on forever. And what would happen if it goes on forever?

So if it goes on forever, let's calculate what this aj plus 2 over aj as j goes to infinity. Let's see how the coefficients vary as you go higher and higher up in the polynomial. That should be an interesting thing.

So I pass the aj that is on the right side and divide it, and now on the right-hand side there's just this product of factors. And as j goes to infinity, it's much larger than 1 or e, whatever it is, and the 2 and the 1 in the denominator. So this goes like 2j over j squared. And this goes roughly like 2 over j.

So as you go higher and higher up, by the time j is a billion, the next term is 2 divided by a billion. And they are decaying, which is good, but they're not decaying fast enough. That's a problem. So let's try to figure out if we know of a function that decays in a similar way.

So you could do it some other way. I'll do it this way. e to the u squared-- let's look at this function-- is this sum from n equals 0 to infinity 1 over n u squared to the n. So it's u to the 2n-- 1 over n factorial, sorry.

So now, since we have j's and they jump by twos, these exponents also here jump by two. So that's about right. So let's think of 2n being j, and therefore this becomes a sum where j equals 0, 2, 4, and all that of 1 over-- so even j's.

2n is equal to j-- so j over 2 factorial over 1, and then you have u to the j. So if I think of this as some coefficient c sub j times u to the j, we've learned that c sub j is equal to 1 over j divided by two factorial. In which case, if that is true, let's try to see what this cj plus 2 over cj-- the ratio of two consecutive coefficients in this series.

Well, cj plus 2 would be j plus 2 over 2 factorial, like this. That's the numerator, because of that formula. And the denominator would have just j over 2 factorial.

Now, these factorials make sense. You don't have to worry that they are factorials of halves, because j is even. And therefore, the numerators are even-- divided by 2. These are integers. These are ordinary factorials.

There are factorials of fractional numbers. You've seen them probably in statistical physics and other fields, but we don't have those here. This is another thing. So this cancels.

If you have a number and the number plus 1, which is here, you get j plus 2 over 2, which is 2 over j plus 2. And that's when j is largest, just 2 over j, which is exactly what we have here. So this supposedly nice, innocent function, polynomial here-- if it doesn't truncate, if this recursive relation keeps producing more and more and more terms forever-- will diverge. And it will diverge like so. If the series does not truncate, h of u will diverge like e to the u squared.

Needless to say, that's a disaster. Because, first, it's kind of interesting to see that here, yes, you have a safety factor, e to the minus u squared over 2. But if h of u diverges like e to the u squared, you're still in trouble. e to the u squared minus u squared over 2 is e to the plus u squared over 2. And it actually coincides with what we learned before, that any solution goes like either plus or minus u squared over 2.

So if h of u doesn't truncate and doesn't become a polynomial, it will diverge like e to the u squared, and this solution will diverge like e to the plus u squared over 2, which was a possibility. And it will not be normalizable. So that's basically the gist of the argument.

This differential equation-- whenever you work with arbitrary energies, there's no reason why the series will stop. Because e there will have to be equal to 2j plus 1, which is an integer. So unless je is an integer, it will not stop, and then you'll have a divergent-- well, not divergent; unbounded-- far of u that is impossible to normalize.

So the requirement that the solution be normalizable quantizes the energy. It's a very nice effect of a differential equation. It's very nice that you can see it without doing numerical experiments, that what's going on here is an absolute requirement that this series terminates. So here, phi of u would go like e to the u squared over 2, what we mentioned there, and it's not a solution.

So if the series must terminate, the numerator on that box equation must be 0 for some value of j, and therefore there must exist a j such that 2j plus 1 is equal to the energy. So basically, what this means is that these unit-free energies must be an odd integer. So in this case, this can be true for j equals 0, 1, 2, 3. In each case, it will terminate the series.

With j equals 0, 1, 2, or 3 there, you get some values of e that the series will terminate. And when this series terminates, aj plus 2 is equal to 0. Because look at your box equation. aj, you got her number, and then suddenly you get this 2j plus 1 minus e.

And if that's 0, the next one is zero. So, yes, you get something interesting even for j equals 0. Because in that case, you can have a0, but you will have no a2, just the constant. So I will write it.

So if aj plus 2 is equal to zero, h of u will be aj u to the j plus aj minus 2u to the j minus 2. And it goes down. The last coefficient that exists is aj, and then you go down by two's.

So let's use the typical notation. We call j equals n, and then the energy is 2n plus 1. The h is an u to the n plus an minus 2. You do the n minus 2, and it goes on.

If n is even, it's an even solution. If n is odd, it's an odd solution. And the energy e, remember, was h omega over 2 times e-- so 2n plus 1. So we'll move the 2 in, and e will be equal to h omega n plus 1/2.

And n in all these solutions goes from 0, 1, 2, 3. We can call this the energy en. So here you see another well-known, famous fact that energy levels are all evenly spaced, h omega over 2, one by one by one-- except that there's even an offset for n equals 0, which is supposed to be the lowest energy state of the oscillator. You still have a 1/2 h bar omega.

This is just saying that if you have the potential, the ground state is already a little bit up. You would expect that-- you know there's no solutions with energy below the lowest point of the potential. But the first solution has to be a little bit up. So it's here and then they're all evenly spaced.

And this begins with E0; for n equals 0, e1. And there's a little bit of notational issues. We used to call the ground state energy sometimes e1, e2, e3, going up, but this time it is very natural to call it E0 because it corresponds to n equals 0. Sorry. Those things happen.

No, it's not an approximation. It's really, in a sense, the following statement. Let me remind everybody of that statement. When you have even or odd solutions, you can produce a solution that you may say it's a superposition, but it will not be an energy eigenstate anymore.

Because the even solution that stops, say, at u to the 6 has some energy, and the odd solution has a different energy. So these are different energy eigenstates. So the energy eigenstates, we prove for one-dimensional potentials, are not chosen to be even or odd for bound states. They are either even or odd.

You see, a superposition-- how do we say like that? Here we have it. If this coefficient is even, the energy sum value-- if this coefficient is odd, the energy will be different. And two energy eigenstates with different energies, the sum is not an energy eigenstate.

You can construct the general solution by superimposing, but that would be general solutions of the full time-dependent Schrodinger equation, not of the energy eigenstates. The equation we're aiming to solve there is a solution for energy eigenstates. And although this concept I can see now from the questions where you're getting, it's a subtle statement.

Our statement was, from quantum mechanics, that when we would solve a symmetric potential, the bound states would turn out to be either even or odd. It's not an approximation. It's not a choice. It's something forced on you.

Each time you find the bound state, it's either even or it's odd, and this turned out to be this case. You would have said the general solution is a superposition, but that's not true. Because if you put a superposition, the energy will truncate one of them but will not truncate the other series. So one will be bad. It will do nothing.

So if this point is not completely clear, please insist later, insist in recitation. Come back to me office hours. This point should be eventually clear. Good.

So what are the names of these things? These are called Hermite polynomials. And so back to the differential equation, let's look at the differential equations when e is equal to 2n plus 1. Go back to the differential equation, and we'll write d second du squared Hn of u.

That will be called the Hermite polynomial, n minus 2udHn du plus e minus 1. But e is 2n plus 1 minus 1 is 2n Hn of u is equal to 0. This is the Hermite's differential equation.

And the Hn's are Hermite polynomials, which, conventionally, for purposes of doing your algebra nicely, people figured out that Hn of u is convenient if-- it begins with u to the n and then it continues down u to the n minus 2 and all these ones here. But here people like it when it's 2 to the n, u to the n-- a normalization. So we know the leading term must be u to the n.

If you truncate with j, you've got u to the j. You truncate with n, you get u to the n. Since this is a linear differential equation, the coefficient in front is your choice. And people's choice has been that one and has been followed.

A few Hermite polynomials, just a list. H0 is just 1. H1 is 2u. H2 is 4u squared minus 2. H3 is our last one, 8u cubed minus 12u, I think. I have a little typo here. Maybe it's wrong. So you want to generate more Hermite polynomials, here is a neat way that is used sometimes.

And these, too, are generating functional. It's very nice actually. You will have in some homework a little discussion.

Look, you put the variable z over there. What is z having to do with anything? u we know, but z, why? Well, z is that formal variable for what is called the generating function. So it's equal to the sum from n equals 0 to infinity. And you expand it kind of like an exponential, zn over n factorial.

But there will be functions of u all over there. If you expand this exponential, you have an infinite series, and then you have to collect terms by powers of z. And if you have a z to the 8, you might have gotten from this to the fourth, but you might have gotten it from this to the 3 and then two factors of this term squared or a cross-product.

So after all here, there will be some function of u, and that function is called the Hermite polynomial. So if you expand this with Mathematica, say, and collect in terms of u, you will generate the Hermite polynomials. With this formula, it's kind of not that difficult to see that the Hermite polynomial begins in this way.

And how do you check this is true? Well, you would have to show that such polynomials satisfy that differential equation, and that's easier than what it seems. It might seem difficult, but it's just a few lines.

Now, I want you to feel comfortable enough with this, so let me wrap it up, the solutions, and remind you, well, you had always u but you cared about x. So u was x over a. So let's look at our wave functions.

Our wave functions phi n of x will be the Hermite polynomial n of u, which is of x over a, times e to the minus u squared over 2, which is minus x squared over 2a squared. And you should remember that a squared is h bar over m omega. So all kinds of funny factors-- in particular, this exponential is e to the minus x squared m omega over h squared over 2. I think so-- m omega over 2h bar.

Let me write it differently-- m omega over 2h bar x squared. That's that exponential, and those are the coefficients. And here there should be a normalization constant, which I will not write.

It's a little messy. And those are the solutions. And the energies en were h bar omega over 2 n plus 1/2, so E0 is equal to h bar omega over 2. E1 is 3/2 of h bar omega, and it just goes on like that.