PROFESSOR: So, what is the wave function that we have? We must have a wave function now that is symmetric, and built with e to the k x, kappa x, and into the minus kappa x. This is the only possibility. E to the minus of kappa absolute value of x. This is psi of x for x different from 0. This is-- as you can quickly see-- this is e to get minus kappa x, when x is positive, A e to the kappa x, when x is negative. And, both of them decay. The first exponential negative is the standard decaying exponential to the right. The one with positive-- well, here x is negative as you go all the way to the left. This one decays case as well.

And, this thing plotted is a decaying exponential with amplitude A, like that. And, a decaying exponential with amplitude A, and a singularity there, which is what you would have expected. So, this seems to be on the right track-- it's a continuous wave function. The wave function cannot fail to be continuous, that's a complete disaster to show that an equation could not be satisfied. So, this is our discontinuous wave function.

So, at this moment you really haven't yet used the delta function-- the delta function with intensity alpha down. I should have made a comment that it's very nice that alpha appeared here in the numerator. If it would have appeared in the denominator, I would be telling you that I think this problem is not going to have a solution. Why? Because if it appears in the numerator, it means that as the delta function potential is becoming stronger and stronger, the bound state is getting deeper and deeper-- which is what you would expect. But, if it would be in the numerator-- in the denominator-- as the potential gets deeper and deeper, the boundary is going up. That makes no sense whatsoever. So, it's good that it appeared there, it's a sign that things are in reasonable conditions.

So, now we really have to face the delta function. And, this is a procedure you are going to do many times in this course. So, look at it, and do it again and again until you're very comfortable with it. It's the issue of discovering what kind of discontinuity you can have with the delta function. And, it's a discontinuity in the derivative, so let's quantify it. So, here it is-- we begin with the Schrodinger equation, again. But, I will write now the potential term as well. The potential is plus v of x psi of x equals E psi of x.

And the idea is to integrate this equation from minus epsilon to epsilon. And, epsilon is supposed to be a small positive number. So, you integrate from minus epsilon to epsilon the differential equation, and see what it does to you in the limit as epsilon goes to 0. That's what we're going to try to do. So, what do we get? If you integrate this, you get minus h squared over 2 m. And now, you have to integrate the second derivative with respect to x, which is the first derivative, and therefore this is the first derivative at x equal epsilon minus the first derivative at x equals minus epsilon. This is from the first term, because you integrate d x d second d x squared psi is the same thing as d x d d x of d psi d x between A and B. And, the integral of a total derivative is d psi d x at B A-- I think people write it like this-- A to B. Evaluate it at the top, minus the evaluation at the bottom.

Now, the next term is the integral of psi times v of x. So, I'll write it plus the integral from minus epsilon to epsilon d x minus alpha delta of x psi of x-- that's the potential. Now, we use the delta function. And, on the right hand side this will be E times the integral from minus epsilon to epsilon of psi of x d x. So, that's the differential equation integrated.

And now, we're going to do two things. We're going to do some of these integrals, and take the limit as epsilon goes to 0. So, I'll write this minus h squared over 2 m limit as epsilon goes to 0 of d psi d x at epsilon minus d psi d x at minus epsilon plus. Let's think of this integral. We can do this integral, it's a delta function. So, it picks the value of the wave function at 0, because 0 is inside the interval of integration. That's why we integrate it from minus epsilon to epsilon, to have the delta function inside. So, you get an alpha out, a psi of 0, and that's what this integral is. It's independent of the value of epsilon as long as epsilon is different from 0. So, this gives you minus alpha psi of 0.

And now, the last term is an integral from minus epsilon to epsilon of the wave function. Now, the wave function is continuous-- it should be continuous-- that means it's finite. And, this integral, as of any function that is not divergent from minus epsilon to epsilon as epsilon goes to 0, is 0. Any integral of a function that doesn't diverge as the limits of integration go to 0, the area under the function is 0. So, this is 0-- the limit. And this thing goes to 0, so we put a 0 here.

So, at this moment we got really what we wanted. I'll write it this way. I'll go here, and I'll say minus h squared over 2 m, and what is this? This expression says, calculate the derivative of the function a little bit to the right of 0, and subtract the derivative of the function a little bit to the left of 0. This is nothing but the discontinuity in psi prime. You're evaluating for any epsilon greater than 0-- the psi prime a little to right, a little too the left, and taking the difference. So, this is what we should call the discontinuity delta at 0-- at x equals 0. And, this and this is for discontinuity of psi prime at x equals 0 minus alpha psi of 0 equals 0. And from here, we discover that delta zero psi prime is equal to minus 2 m alpha over h squared psi of 0.

This is the discontinuity condition produced by the delta function. This whole quantity is what we call delta 0 of psi prime. And, what it says is that yes, the wave function can have a discontinuous first derivative if the wave function doesn't vanish there. Once the wave function doesn't vanish at that point, the discontinuity is in fact even proportional to the value of the wave function at that point. And, here are the constants of proportionality. Now, I don't think it's worth to memorize this equation or anything like that, because it basically can be derived in a few lines. This may have looked like an interesting or somewhat intricate derivation, but after you've done is a couple of times-- this is something you'll do in a minute or so. And, you just integrate and find the discontinuity in the derivative-- that's a formula there. And, that's a formula for a potential, minus alpha delta of x. So, if somebody gives you a different potential, well, you have to change the alpha accordingly.

So, let's wrap this up. So, we go to our case. Here is our situation. So, let's apply this. So, what is the value? Apply this equation to our wave function. So, what is the derivative at epsilon? It's minus kappa A E to the minus kappa epsilon. That's the derivative of psi on the positive side. I differentiated the top line of this equation minus the derivative on the left side-- this one, the derivative. So, this is kappa A E to the kappa epsilon-- no, kappa minus epsilon again. So, that's the left hand side. The right hand side would be minus 2 m alpha h squared psi at 0. Psi at 0 is A, so that's what it gives us.

And we should take the limit as epsilon goes to 0. So, this is going to 1, both of them. So, the left hand side is minus 2 kappa A, and the right hand side is 2 m alpha over h squared A. So, the 2-- it's also minus, I'm sorry-- so the 2s cancel, the A cancels-- you never should have expected to determine A unless you tried to normalize the wave function. Solving for energy eigenstates states will never determine A. The Schrodinger equation is linear, so A drops out, the minus 2 drops out, and kappa is equal to m alpha over h squared.

So, that said that's great because kappa is just another name for the energy. So, I have kappa m alpha over h squared, so that's another name for the energy. So, let's go to the energy. The energy is h bar squared kappa minus h bar squared kappa squared over 2 m. So, it's minus h bar squared. Kappa squared would be m squared alpha squared h to the fourth, and there's a two m. All these constants.

So, final answer. E, the bound state energy is minus m alpha squared minus m alpha squared.

The m cancels it over h squared minus one half. So, back here the units worked out, everything is good, and the number was determined as minus one half. That's your bound state energy for this problem.

So, this problem is instructive because you basically learn that in delta functions, with one delta function you get a bound state. If you have two delta functions, you may get more bound states-- three, four-- people study those problems, and you will investigate the two delta function cases.