Quantum Physics I (8.04) Spring 2016
Assignment 5

MIT Physics Department — *Due Fri. March 18, 2016*

March 11, 2015 — *12:00 noon*

Reading: Griffiths: 3.2, 3.3, 3.4, 2.1 and 2.2.

Problem Set 5

1. **Gaussians and uncertainty product saturation** [5 points]

   Consider the gaussian wavefunction

$$\psi(x) \;=\; N \exp\Big(-\frac{1}{2}\frac{x^2}{a^2}\Big)\,, \tag{1}$$

   where $N \in \mathbb{R}$ a is a real positive constant with units of length. The integrals

$$\begin{aligned}\int_{-\infty}^{\infty} dx e^{-\alpha x^2+\beta x} \;&=\; \sqrt{\frac{\pi}{\alpha}}\,\exp\Big(\frac{\beta^2}{4\alpha}\Big)\,, \quad \text{Re}(\alpha)>0\,,\\ \int_{-\infty}^{\infty} dx\, x^2 e^{-\alpha x^2} \;&=\; \frac{1}{2\alpha}\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}\end{aligned}$$

   (a) Use the position space wavefunction (1) to calculate the uncertainties Δx and Δp. Confirm that your answer saturates the Heisenberg uncertainty product

$$\Delta x \Delta p \geq \tfrac{\hbar}{2}\,.$$

   (Hints: These calculations are actually quite brief if done the right way! Using the second of the above integrals you don't even have to determine N. For the evaluation of $\langle \hat{p}^2\rangle$ in position space fold one factor of $\hat{p}$ into ψ^*.).

   (b) Calculate the Fourier transform $\phi(p)$ of $\psi(x)$. Use Parseval to confirm your answer and then recalculate Δp using momentum space.

2. **Complex Gaussians and the uncertainty product** [10 points]

   Consider the gaussian wavefunction

$$\psi(x) \;=\; N \exp\Big(-\frac{1}{2}\frac{x^2}{\Delta^2}\Big)\,, \quad \Delta \in \mathbb{C}\,, \quad \text{Re}(\Delta^2)>0\,, \tag{1}$$

   where N is a real normalization constant and Δ is now a complex number: $\Delta^* \neq \Delta$. The integrals in Problem 1 are also useful here and so is the following relation, valid for any nonzero complex number z,

$$\text{Re}\Big(\frac{1}{z}\Big) \;=\; \frac{\text{Re}(z)}{|z|^2} \qquad \text{(prove it!)}$$

(a) Use the position space representation (1) of the wavefunction to calculate the uncertainties Δx and Δp. Leave your answer in terms of $|\Delta|$ and $\text{Re}(\Delta^2)$. (Δx will depend on both[1], while Δp will depend only on $\text{Re}(\Delta^2)$).

(b) Calculate the Fourier transform $\phi(p)$ of $\psi(x)$. Use Parseval to confirm your answer and then recalculate Δp using momentum space.

(c) We parameterize Δ using a phase $\phi_\Delta \in \mathbb{R}$ as follows

$$\Delta \;=\; |\Delta|\, e^{i\phi_\Delta}\,.$$

Calculate the product $\Delta x \Delta p$ and confirm that the answer can be put in terms of a trigonometric function of ϕ_Δ and that $|\Delta|$ drops out. Is your answer reasonable for $\phi_\Delta = 0$ and for $\phi_\Delta = \frac{\pi}{4}$?

(d) Consider the free evolution of a gaussian wave packet in Problem 3 of Homework 4. What is Δp at time equal zero? Examine the time evolution of the gaussian (from the solution!) and read the value of the time-dependent (complex) constant Δ^2. Confirm that Δp, found in (a), gives a time-independent result.

3. **Exercises with a particle in a box** [15 points]

Consider a 1D problem for a particle of mass m that is free to move in the interval $x \in [0, a]$. The potential $V(x)$ is zero in this interval and infinite elsewhere. For that system consider a solution of the Schrödinger equation of the form

$$\Psi_n(x,t) \;=\; N \sin\Big(\frac{n\pi}{a}x\Big) e^{-i\phi_n(t)}\,, \qquad x \in [0,a]\,,$$

and $\Psi_n(x,t) = 0$ for $x < 0$ and $x > a$. Here $n \geq 1$ is an integer.

(a) Find the expression for the (real) phase $\phi_n(t)$ so that the above wavefunction solves the Schrödinger equation. Find the normalization constant N.

(b) Use $\Psi_n(x,0)$ to calculate $\langle x\rangle$, $\langle x^2\rangle$, and Δx.

(c) Use $\Psi_n(x,0)$ to calculate $\langle p\rangle$, $\langle p^2\rangle$, and Δp.

(d) Is the uncertainty inequality satisfied? Is it saturated?

(e) What answers in (b) and (c) change for $\Psi_n(x,t)$? Explain.

4. **A Hard Wall** [5 points]

A particle of mass m is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} 0, & \text{for } x > 0, \\ \infty & \text{for } x \leq 0\,. \end{cases}$$

Find the stationary states and their energies. These states cannot be normalized.

[1] Actually Δx can be written in terms of $\text{Re}(1/\Delta^2)$ alone.

5. **A Step Up on the Infinite Line** [10 points]

   A particle of mass m is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} V_0, & \text{for } x > 0, \\ 0\,, & \text{for } x \leq 0\,. \end{cases}$$

   Find the stationary states that exist for energies $0 < E < V_0$.

6. **A Wall and Half of a Finite Well** [10 points]

   A particle of mass m is moving in one dimension, subject to the potential $V(x)$:

$$V(x) = \begin{cases} \infty\,, & \text{for } x < 0, \\ -V_0, & \text{for } 0 < x < a\,, \quad (V_0 > 0) \\ 0\,, & \text{for } x > a\,. \end{cases}$$

   Find the stationary states that correspond to bound states ($E < 0$, in this case). Is there always a bound state? Find the minimum value of z_0

$$z_0^2 = \frac{2ma^2V_0}{\hbar^2}\,,$$

   for which there are three bound states. Explain the precise relation of this problem to the problem of the finite square well of width $2a$.

7. **Mimicking hydrogen with a one-dimensional square well.** [5 points]

   The hydrogen atom the Bohr radius a_0 and ground state energy E_0 are given by

$$a_0 = \frac{\hbar^2}{me^2} \simeq 0.529 \times 10^{-10}\text{m}\,, \quad E_0 = -\frac{e^2}{2a_0} = -13.6 \text{ eV}.$$

   The ground state is a bound state and the potential goes to zero at infinity. We want to design a one-dimensional finite square well

$$V(x) = \begin{cases} -V_0, & \text{for } |x| < a_0, \quad V_0 > 0\,, \\ 0\,, & \text{for } |x| > a_0\,, \end{cases}$$

   that simulates the hydrogen atom. Calculate the value of V_0 in eV so that the ground state of the box is at the correct depth.

8. **No states with $E < V(x)$** [5 points]

   Consider a real stationary state $\psi(x)$ with energy E:

$$-\frac{\hbar^2}{2m}\psi''(x) + [V(x) - E]\psi(x) \;=\; 0\,.$$

   (a) Prove that E must exceed the minimum value of $V(x)$ by noting that $E = \langle H \rangle$.

   (b) Explain the claim by trying (and failing) to sketch a wavefunction consistent with being on the classically inaccessible region for all values of x.