

Quantum Physics I (8.04) Spring 2016

Assignment 4

MIT Physics Department
February 25, 2016

Due Fri. March 4, 2016
12:00 noon

Announcements

- Recommended Reading: Griffiths, sections 1.5, 1.6, 2.4.

Problem Set 4

1. Exercises on packets changing shape [5 points]

- (a) A free proton is localized within $\Delta x = 10^{-10}$ m. Estimate the time t_s it takes the packet to spread appreciably. Repeat for an proton localized within 1 cm.
- (b) Consider a wave packet that satisfies the relation $\Delta x \Delta p \sim \hbar$. Show that the condition $\Delta p \ll p$ guarantees that the packet does not spread appreciably in the time it takes to pass through a fixed position.

2. Probability current in three dimensions [10 points]

In elastic scattering of particles in three-dimensional space the wavefunction takes the form

$$\Psi(\mathbf{x}) = e^{ikz} + \frac{f(\theta)}{r} e^{ikr}, \quad \text{valid for large } r.$$

The time dependence has been suppressed; it is just an overall time dependent phase $e^{-iEt/\hbar}$ with $E = \hbar^2 k^2 / (2m)$. It will play no role here.

The first term represents the incoming particles, moving in the $+z$ direction. The target is located at the origin $r = 0$ and the second term represents the amplitude for particles moving radially out – the scattered particles. This amplitude is θ dependent but assumed to be ϕ independent; $f(\theta)$ is a complex function of θ that carries the information about the scattering. Recall that θ is the polar angle and $z = r \cos \theta$.

When you calculate the probability current $\mathbf{J}(\mathbf{x})$ associated with Ψ there will be a contribution \mathbf{J}_1 due to the first term (the plane wave), a contribution \mathbf{J}_2 due to the second term (the spherical waves), and a contribution \mathbf{J}_{12} due to interference between the first and second term:

$$\mathbf{J}(\mathbf{x}) = \mathbf{J}_1(\mathbf{x}) + \mathbf{J}_2(\mathbf{x}) + \mathbf{J}_{12}(\mathbf{x}).$$

- (a) Calculate the probability current \mathbf{J}_1 and the total flux of this current over a large sphere of radius R centered at the origin $r = 0$.

- (b) Calculate the radial component $\hat{\mathbf{r}} \cdot \mathbf{J}_2$ of the probability current \mathbf{J}_2 . Here $\hat{\mathbf{r}}$ is the unit vector in the radial direction. Calculate the flux of this current over a radius R sphere centered at the origin in the limit as $R \rightarrow \infty$. Your answer should be left as an integral over solid angle $\int d\Omega$, or over $\int d\theta$, if you prefer.
- (c) Calculate the *radial* component of the interference term \mathbf{J}_{12} but pick up only the leading part in $1/r$ (that is ignore $1/r^2$ terms). Show that the answer can be written in the form

$$\hat{\mathbf{r}} \cdot \mathbf{J}_{12} = \frac{\hbar k}{mr} \text{Im}[i(\dots)]$$

where (\dots) represent terms that your calculation should determine. These terms depend on $f(\theta)$, $\cos \theta$, and the product kr in exponentials. Calculating the flux of this current over the large sphere is delicate so we will leave that for later (the end result is the so called *optical theorem!*).

3. Evolving the Gaussian wave packet [15 points]

Consider the normalized wave packet representing the state of a particle of mass m at $t = 0$:

$$\Psi_a(x, 0) = \frac{1}{(2\pi)^{1/4} \sqrt{a}} \exp\left(-\frac{x^2}{4a^2}\right).$$

Here a is a length parameter that represents the width of the packet at zero time.

- (a) Confirm that $\Psi_a(x, 0)$ is properly normalized.
- (b) Find the Fourier representation of $\Psi_a(x, 0)$, namely, determine the function $\Phi_a(k)$ such that

$$\Psi_a(x, 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_a(k) e^{ikx} dk.$$

- (c) Assume the particle is free to find the wavefunction $\Psi_a(x, t)$ for arbitrary $t > 0$. The answer is a bit messy, but can be written more clearly using the time constant τ built from the constants in the problem:

$$\tau \equiv \frac{2ma^2}{\hbar}.$$

- (d) At time zero the probability density is

$$|\Psi_a(x, 0)|^2 = \frac{1}{\sqrt{2\pi} a} \exp\left(-\frac{x^2}{2a^2}\right) \equiv G(x; a),$$

where we defined the gaussian $G(x; a)$ with width parameter a . What is the probability density $|\Psi_a(x, t)|^2$ for $t > 0$. Express your answer in terms of the gaussian G with a time-dependent width parameter $a(t)$. Give $a(t)$.

Useful integral: Valid for complex constants a and b , with real part of a positive:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right), \quad \text{when } \text{Re}(a) > 0.$$

4. **Parseval's identity in 1D and 3D, and application** [10 points]

- (a) Consider the Fourier pair $(\Psi(x), \Phi(p))$ relevant to one dimensional (1D) wavefunctions and the Fourier pair $(\Psi(\mathbf{x}), \Phi(\mathbf{p}))$ relevant to three-dimensional (3D) wavefunctions. Use the Fourier relations and the integral form for the delta function to prove the 1D and 3D versions of Parseval's identity:¹

$$\int_{-\infty}^{\infty} dx |\Psi(x)|^2 = \int_{-\infty}^{\infty} dp |\Phi(p)|^2,$$

$$\int d^3x |\Psi(\mathbf{x})|^2 = \int d^3p |\Phi(\mathbf{p})|^2.$$

- (b) In the hydrogen atom the ground state wavefunction takes the form $\Psi(\mathbf{x}) = Ne^{-r/a_0}$ where $r = |\mathbf{x}|$, a_0 is the Bohr radius, and N is a normalization constant. Find N . The Fourier transform (which you need not derive) takes the form

$$\Phi(\mathbf{p}) = \frac{N'}{\left(1 + \frac{a_0^2 p^2}{\hbar^2}\right)^2},$$

for some constant N' and with $p \equiv |\mathbf{p}|$. Find N' (you may use an algebraic manipulator to do the integral). Calculate the probability that the electron may be found with a momentum whose magnitude exceeds \hbar/a_0 . (Write your integrals explicitly, but you may evaluate them with a computer). [The momentum distribution was measured by ionization of atomic hydrogen by a high energy electron beam, see, Lohan, B. and Weigold, E. (1981) "Direct measurement of the Electron Momentum Probability Distribution in Atomic Hydrogen," Phys. Lett. **86A**, 139-141.]

5. **Ehrenfest theorem** [10 points]

Consider a particle moving in one dimension with Hamiltonian H given by

$$H = \frac{p^2}{2m} + V(x).$$

Show that the expectation values $\langle x \rangle$ and $\langle p \rangle$ are time-dependent functions that satisfy the following differential equations:

$$\frac{d}{dt} \langle x \rangle = \frac{1}{m} \langle p \rangle,$$

$$\frac{d}{dt} \langle p \rangle = - \left\langle \frac{\partial V}{\partial x} \right\rangle.$$

¹In mathematics this is called Plancherel's theorem. Parseval's result was the analog for Fourier series.

6. Momentum uncertainty [5 points]

Show that in a free-particle wave packet the momentum uncertainty Δp does not change in time.

7. Finding Meaning in the Phase of the Wavefunction [10 points]

Suppose $\psi_o(x)$ is a properly-normalized wavefunction with $\langle x \rangle_{\psi_o} = x_o$ and $\langle p \rangle_{\psi_o} = p_o$, where x_o and p_o are constants. Define the boost operator \hat{B}_q to be the operator that acts on arbitrary functions of x by multiplication by a q -dependent phase:

$$\hat{B}_q f(x) = e^{iqx/\hbar} f(x).$$

Here q is a real number with the appropriate units. Consider now a new wavefunction obtained by boosting the initial wavefunction:

$$\psi_{new}(x) = \hat{B}_q \psi_o(x).$$

- What is the expectation value $\langle x \rangle_{\psi_{new}}$ in the state given by $\psi_{new}(x)$?
- What is the expectation value $\langle p \rangle_{\psi_{new}}$ in the state given by $\psi_{new}(x)$?
- Based on your results, what is the physical significance of adding an overall factor $e^{iqx/\hbar}$ to a wavefunction.
- Compute $[\hat{p}, \hat{B}_q]$ and $[\hat{x}, \hat{B}_q]$.

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