Problem Set 4

Due Tuesday March 5 at 11.00AM

Assigned Reading:

E&R 5_{all}, $6_{1,2,8}$
Li. 3_{all}, 4_1, 5_1, 6_{all}
Ga. 2_4, 3_{all}
Sh. 4_{all}, $5_{1,2}$

1. **(10 points) Simultaneous Eigenstates**

   Your friend from last year's 8.04 argues the following: If a particle is in an eigenstate of a one-dimensional box of width L, then we know its energy exactly. We also know that the energy in the box is purely kinetic. Hence we know the particle's momentum exactly as well. This contradicts the Heisenberg uncertainty relation since the uncertainty in the particle position is finite ($\Delta x < L$). Punch a hole into your friend's argument.

2. **(10 points) Formal Properties of Energy Eigenstates**

   Give a physicist's proof of the following statements regarding energy eigenfunctions:

   (a) We can always choose the energy eigenstates $\phi_E(x)$ we work with to be purely <u>real</u> functions (unlike the physical wavefunction $\psi(x,t)$, which is necessarily complex).[1]

   Note: This does not mean that every energy eigenfunction is real – rather, if you find an eigenfunction that is not real, it can always be written as a complex linear combination of two real eigenstates with the same energy.

   (b) If $V(x)$ is an even function [*i.e.* $V(-x) = V(x)$], then the energy eigenfunctions $\phi_E(x)$ can always be taken to be <u>either even or odd</u>.[2]

   (c) The lowest energy eigenvalue, E_0, corresponding to a normalizable eigenfunction is strictly greater then the minimum value V_{min} of the potential, $V(x)$, *i.e.*[3]

   $$\boxed{E_0 > V_{min}} \ .$$

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[1] *Hint: If $\phi_E(x)$ is an energy eigenstate with energy eigenvalue E, what can be said about $\phi_E(x)^*$?*
[2] *Hint: If $\phi_E(x)$ is an $\hat{E}$ eigenstate with energy E, what can be said about $\phi_E(-x)$?*
[3] *Hint: Rewrite the eigen-equation as $\frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2}[V(x) - E]\psi(x)$. What happens if $E < V_{min}$?*

3. **(20 points) Superposition in the Infinite Well**

Verify (on scratch paper, no need to turn this in) the results stated in lecture for the eigenvalues of the energy operator for the infinite potential well of width L,

$$V(x) = \left\{ \begin{array}{cc} 0 & 0 \le x \le L \\ \infty & else \end{array} \right\}$$

i.e. remind yourself why the energy eigenvalues are,

$$\boxed{E_n = \frac{{}^2 k_n^2}{2m} \qquad k_n = \frac{(n+1)\pi}{L}} .$$

Now suppose that at $t = 0$ we place a particle in an infinite well in the state

$$\psi_A(x, 0) = \sqrt{\frac{1}{6}}\phi_0(x) + \sqrt{\frac{1}{3}}\phi_1(x) + \sqrt{\frac{1}{2}}\phi_2(x) .$$

Note: Each step below requires relatively little computation. You will not need the functional form of the energy eigenfunctions $\phi_n(x)$ to complete them, only the energy eigenvalues.

(a) How does ψ_A evolve with time? Write down the expression for $\psi_A(x, t)$.

(b) Calculate the expectation value of the energy, $\langle \hat{E} \rangle$, for the particle described by $\psi_A(x, t)$. Write your answer in terms of E_0. Does this quantity change with time?

(c) What is the probability of measuring the energy to equal $\langle \hat{E} \rangle$ as a result of a single measurement at $t=0$? At a later time $t=t_1$?

(d) What energy values will be observed as a result of a single measurement at $t=0$ and with what probabilities? How do these probabilities change with time?

(e) The energy of the particle is found to be E_2 as a result of a single measurement at $t=t_1$. Write down the wave function $\psi_A(x, t)$ which describes the state of the particle for $t>t_1$. What energy values will be observed and with what probabilities at a time $t_2 > t_1$?

(f) Construct another normalized wave function $\psi_B(x, 0)$ which is linearly independent of $\psi_A(x, 0)$, but yields the same value of $\langle \hat{E} \rangle$ as well as the same set of measured energies with the same probabilities.

(g) Construct another normalized wave function $\psi_C(x, 0)$ which is linearly independent of $\psi_A(x, 0)$, yields the same value of $\langle \hat{E} \rangle$, but allows a different set of measured energies (which may include some but not all of E_0, E_1 and E_2, plus others).

4. **(25 points) "Sloshing" Superposition State in the Infinite Potential Well**

   Consider a particle of mass m that is in a superposition state of the first two eigenstates of an infinite potential well of width L,

 $$\psi(x,t) = \sqrt{\frac{1}{L}} \sin\left(\frac{\pi}{L}x\right) e^{-i\omega_0 t} + \sqrt{\frac{1}{L}} \sin\left(\frac{2\pi}{L}x\right) e^{-i\omega_1 t}$$

   for $0 \leq x \leq L$.

   (a) Play around with the PhET Quantum Bound States Applet for insight into this system. Specifically, build the above superposition state and watch the system evolve. Print a screenshot of this superposition and include it in your pset. Experiment with other superpositions, too!

   (b) Verify that $\psi(x,t)$ is properly normalized and remains so for all time t.

   (c) Calculate the probability distribution $\mathbb{P}(x,t) = |\psi(x,t)|^2$. What is the period T of this superposition – *i.e.*, after what time T will the system return to its original configuration?

   (d) Sketch the probability distribution $\mathbb{P}(x,t)$ evaluated at time $t_* = \pi / [2(E_1 - E_0)]$. What fraction of T is t_*?

   (e) What's the probability that you find the particle in the left half of the well at some arbitrary time t?

   (f) Find the expectation value $\langle x \rangle$ of the particle's position as a function of time.

   (g) Show that the probability density $\mathbb{P}(x,t)$ at $x = L/2$ is independent of time.

   (h) You've shown that the probability density at the center $x = L/2$ is time-independent. However, the rest of the probability distribution for the particle sloshes back and forth between the left and right halves of the well. Briefly discuss how this occurs. Identify a modification of the initial superposition that would change this fact.

5. **(35 points) Qualitative Properties of Energy Eigenstates**

   (a) Sketch qualitatively the indicated energy eigenstates of the following potential:
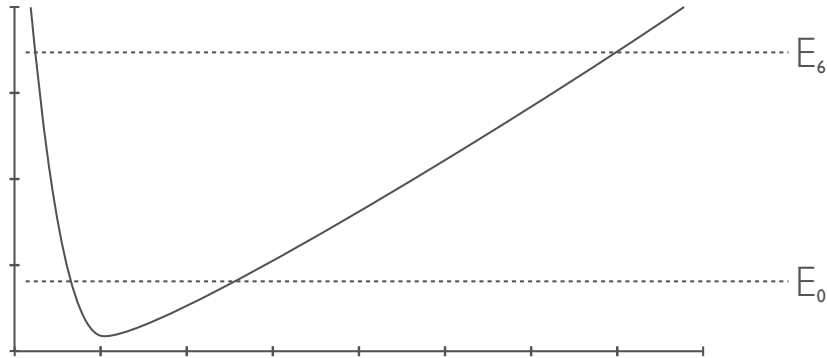
   (b) Sketch qualitatively the indicated energy eigenstates of the following potential:
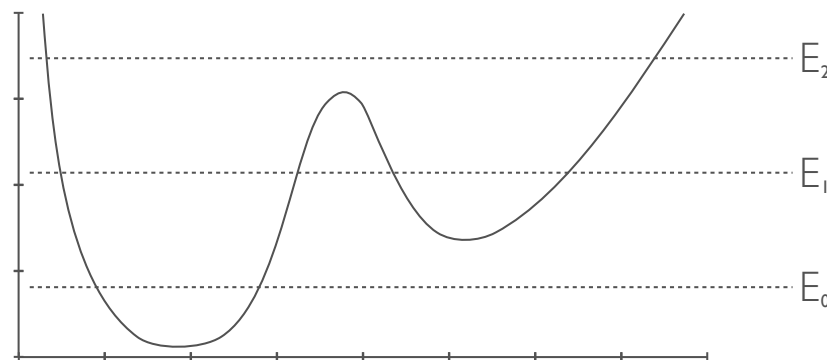
   (c) Sketch qualitatively the indicated eigenstates of the following potential when the height of the central barrier is (i) $V_0 = 0$, (ii) $E_1 < V_0 < E_2$, and (iii) $E_2 \ll V_0$:
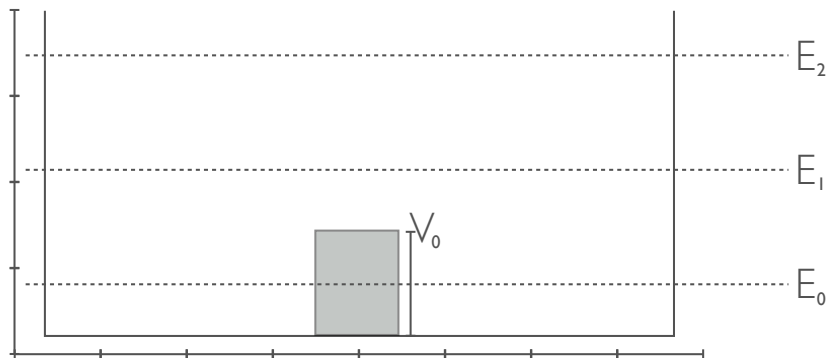
   For $V_0 \to \infty$, the first two eigenstates (with energies E_0 and E_1) are well approximated by superpositions of the ground states of each well. Is the ground state of the double-well system symmetric or anti-symmetric? Why? Do you expect this to be a general property of ground states? *Very Useful Hint: Play around with the* PhET Double Well Java Applet *for insight into this phenomenon!*

8.04 Quantum Physics I

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