Summary : Solution of $\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = 0$

$\boxed{(0)}$ $\Gamma = 0$ No damping

$$\theta(t) = A\cos(\omega_0 t + \alpha)$$

(1) $\omega_0^2 > \frac{\Gamma^2}{4}$ Underdamped Oscillator

$$\theta(t) = A e^{-\frac{\Gamma}{2}t}\cos(\omega t + \alpha) \qquad \omega = \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

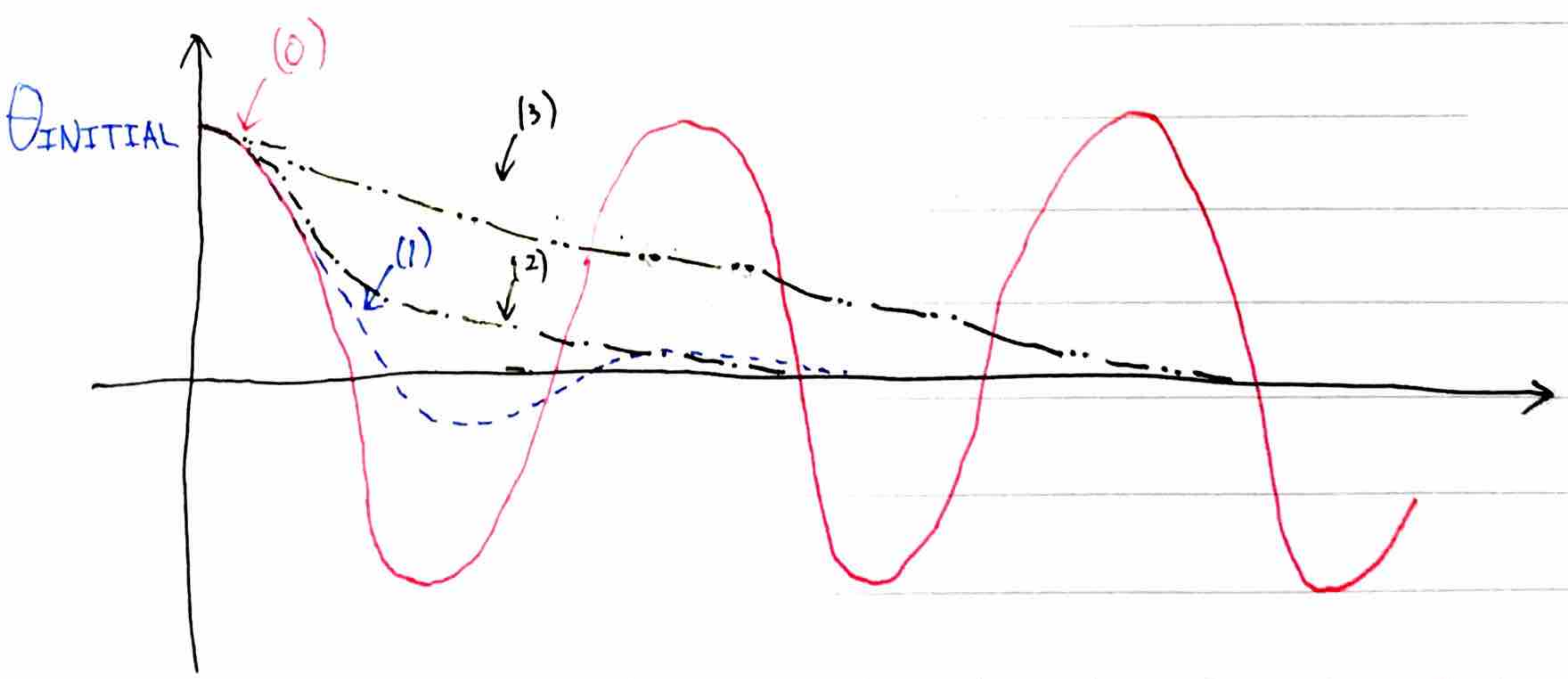
(2) $\omega_0^2 = \frac{\Gamma^2}{4}$ Critically damped Oscillator

$$\theta(t) = (A + Bt)e^{-\frac{\Gamma}{2}t}$$

(3) $\omega_0^2 < \frac{\Gamma^2}{4}$ Overdamped

$$\theta(t) = A e^{-(\frac{\Gamma}{2}+\beta)t} + B e^{-(\frac{\Gamma}{2}-\beta)t}$$

$$\beta = \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$$



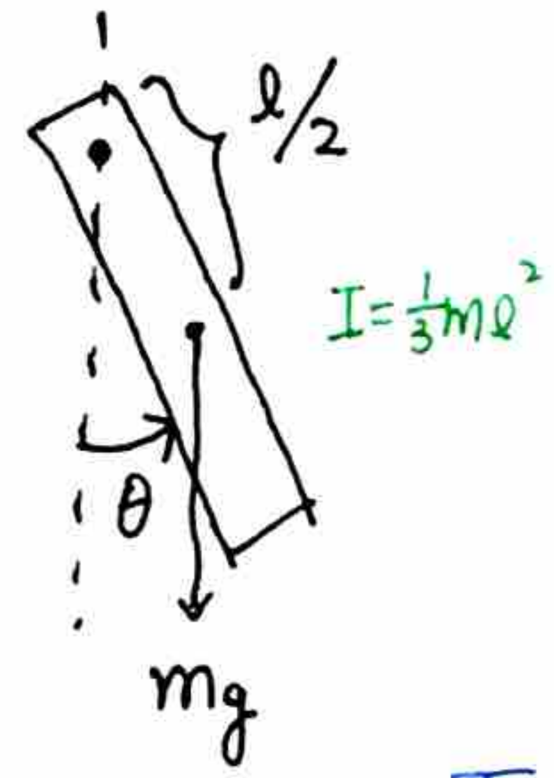
Slide 5

Continue from lecture 2

Now we are interested in giving a driving force to this rod:

Assume that force produce a torque:

$$\tau_{DRIVE} = d_0 \cos \omega_D t$$



Total torque:

$$\tau(t) = \tau_g(t) + \tau_{DRAG}(t) + \tau_{DRIVE}(t)$$

E.O.M: $$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = \frac{d_0}{I}\cos\omega_d t$$

$\Gamma \equiv \dfrac{3b}{ml^2}$, $\omega_0 \equiv \sqrt{\dfrac{3g}{2l}}$ (from previous lecture)

$\Downarrow$ "size of the drag force"

$\Downarrow$ "nature angular frequency"

Define $f_0 \equiv \dfrac{d_0}{I}$

$$\Rightarrow \quad \ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = f_0 \cos\omega_d t$$

SL7

Q: oscillation frequency?

We could like to construct something to "cancel" $\cos\omega_d t$

Idea: use complex notation:

$$\Rightarrow \quad \ddot{z} + \Gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_d t}$$

Guess: $Z(t) = A e^{i(\omega_d t - \delta)}$

Designed to cancel $e^{i\omega_d t}$

It takes some time for the system to "feel" the driving torque.

$$\dot{Z}(t) = i\omega_d Z$$

$$\ddot{Z}(t) = -\omega_d^2 Z$$

Insert those results to the equation of motion:

$$\left[-\omega_d^2 + i\omega_d \Gamma + \omega_0^2\right] Z = f_0 e^{i\omega_d t}$$

$$\left[-\omega_d^2 + i\omega_d \Gamma + \omega_0^2\right] A e^{i(\cancel{\omega_d t} - \delta)} = f_0 e^{\cancel{i\omega_d t}}$$

$\Rightarrow$ We arrive this expression:

$$\left[-\omega_d^2 + i\omega_d \Gamma + \omega_0^2\right] A = f e^{i\delta}$$

$$= f(\cos\delta + i\sin\delta)$$

Since this is a complex equation,
We can solve A and δ:

Real Part: $(\omega_0^2 - \omega_d^2) A = f_0 \cos\delta$ ---(1)

Imaginary Part: $\omega_d \Gamma A = f_0 \sin\delta$ ---(2)

$(1)^2 + (2)^2$: $A^2 [(\omega_0^2 - \omega_d^2)^2 + \omega_d^2 \Gamma^2] = f_0^2$

$$A(\omega_d) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2 \Gamma^2}}$$

$\frac{(2)}{(1)}$: $\tan\delta = \frac{\Gamma \omega_d}{\omega_0^2 - \omega_d^2}$

$\Rightarrow \theta(t) = Re(Z(t)) = A(\omega_d) \cos(\omega_d t - \delta(\omega_d))$

Decided by ω_d Decided by ω_d

No free parameter ?! Actually this is particular solution.

The full solution is (if we prepare the system in "under damped" mode)

$$\omega = \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

$$\theta(t) = \underbrace{A(\omega_d) \cos(\omega_d t - \delta)}_{\text{Steady State Solution}} + \underbrace{B e^{-\frac{\Gamma}{2}t} \cos(\omega t + \alpha)}_{\text{Will die out as } t \to \infty}$$

slide 8

You may be confused : so many different ω ?!

ω_0 : "Natural angular frequency"

In this case: $\omega_0 = \sqrt{\frac{3g}{2l}}$

ω : frequency is Lower if there is drag force

$$\omega = \sqrt{\omega_0^2 - \Gamma^2/4}$$

ω_d : frequency of the driving torque or force.

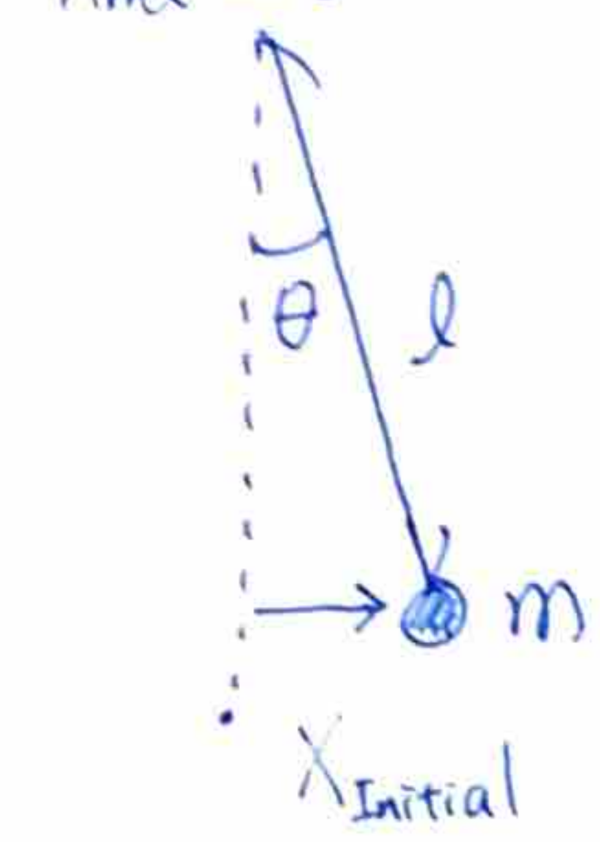
Demo Cart.

(1) Natural frequency (Driving force off)

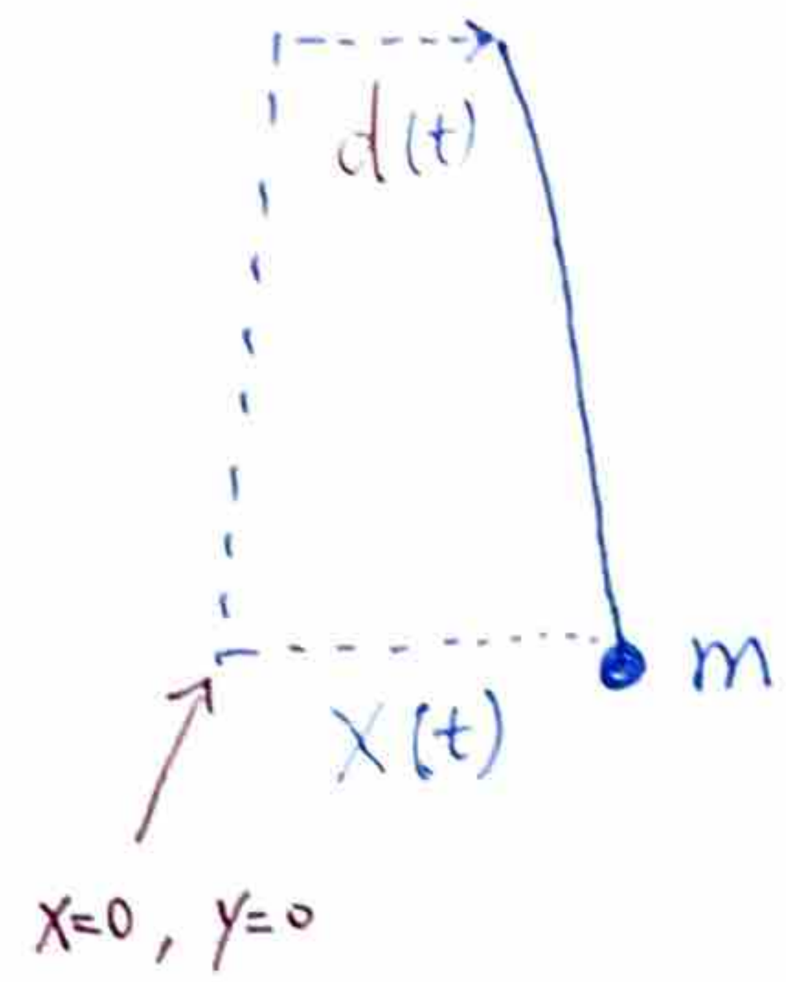
(2) Demo transient behavior (Driving force on)

Drive a pendulum:

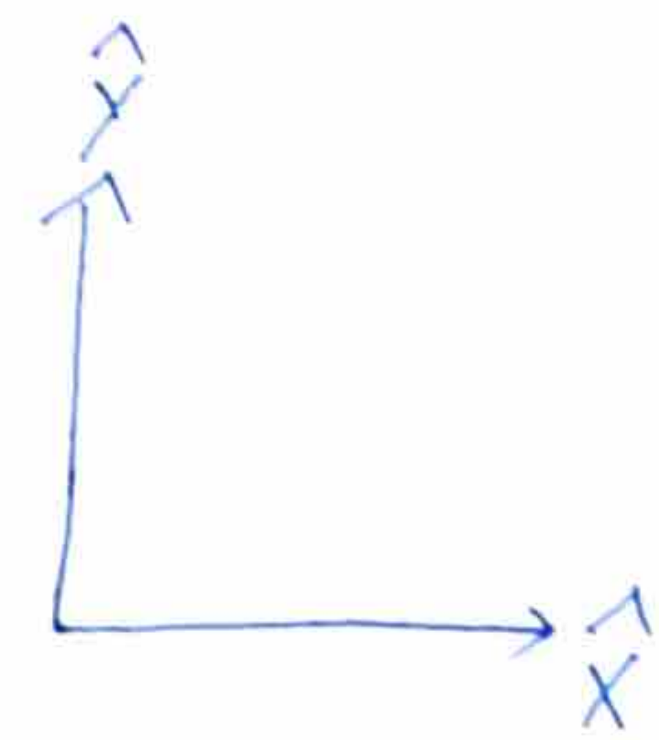
Time = 0



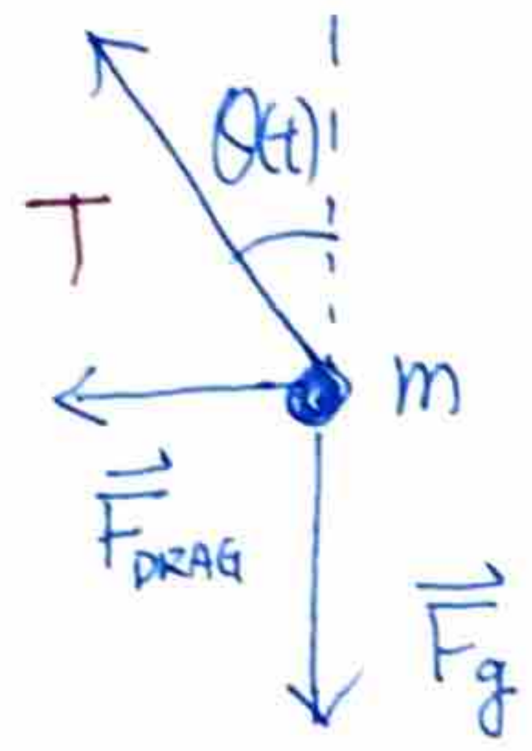
time = t



$d(t) = \Delta \sin \omega_d t$



Force Diagram:



$$\vec{F}_{DRAG} = -b\dot{x}\ \hat{x}$$

$$\vec{F}_g = -mg\ \hat{y}$$

$$\vec{T} = -T\sin\theta\ \hat{x} + T\cos\theta\ \hat{y}$$

Take small angle approximation:

$$\sin\theta \approx \theta = \frac{x-d}{\ell}$$

$$\cos\theta \approx 1$$

$$\Rightarrow \quad \vec{T} \approx -T\frac{(x-d)}{\ell}\hat{x} + T\hat{y}$$

$$\hat{x}: \quad m\ddot{x} = -b\dot{x} - T\frac{(x-d)}{\ell}$$

$$\hat{y}: \quad m\ddot{y} = -mg + T$$

$\|$ (small angle)

0

No vertical motion

$\Rightarrow T = mg$

$$\Rightarrow \quad m\ddot{x} + b\dot{x} + \frac{mg}{\ell}x = \frac{mg}{\ell}d = \frac{mg}{\ell}\Delta \sin W_d t$$

$1/m$

$$\Rightarrow \quad \ddot{x} + \underbrace{\frac{b}{m}}_{\equiv T}\dot{x} + \underbrace{\frac{g}{\ell}}_{\equiv W_0^2}x = \underbrace{\frac{g\Delta}{\ell}}_{\equiv f_0}\sin W_d t$$

$$\ddot{x} + T\dot{x} + W_0^2 x = f_0 \sin W_d t$$

$$A(W_d) = \frac{f_0}{\sqrt{(W_0^2 - W_d^2)^2 + W_d^2 T^2}}$$

(1) $W_d \to 0$

① $$A(W_d) = \frac{f_0}{W_0^2} = \frac{\frac{g\Delta}{\ell}}{g/\ell} = \Delta$$

The amplitude will be equal to the amplitude of Yen-Jie's hand

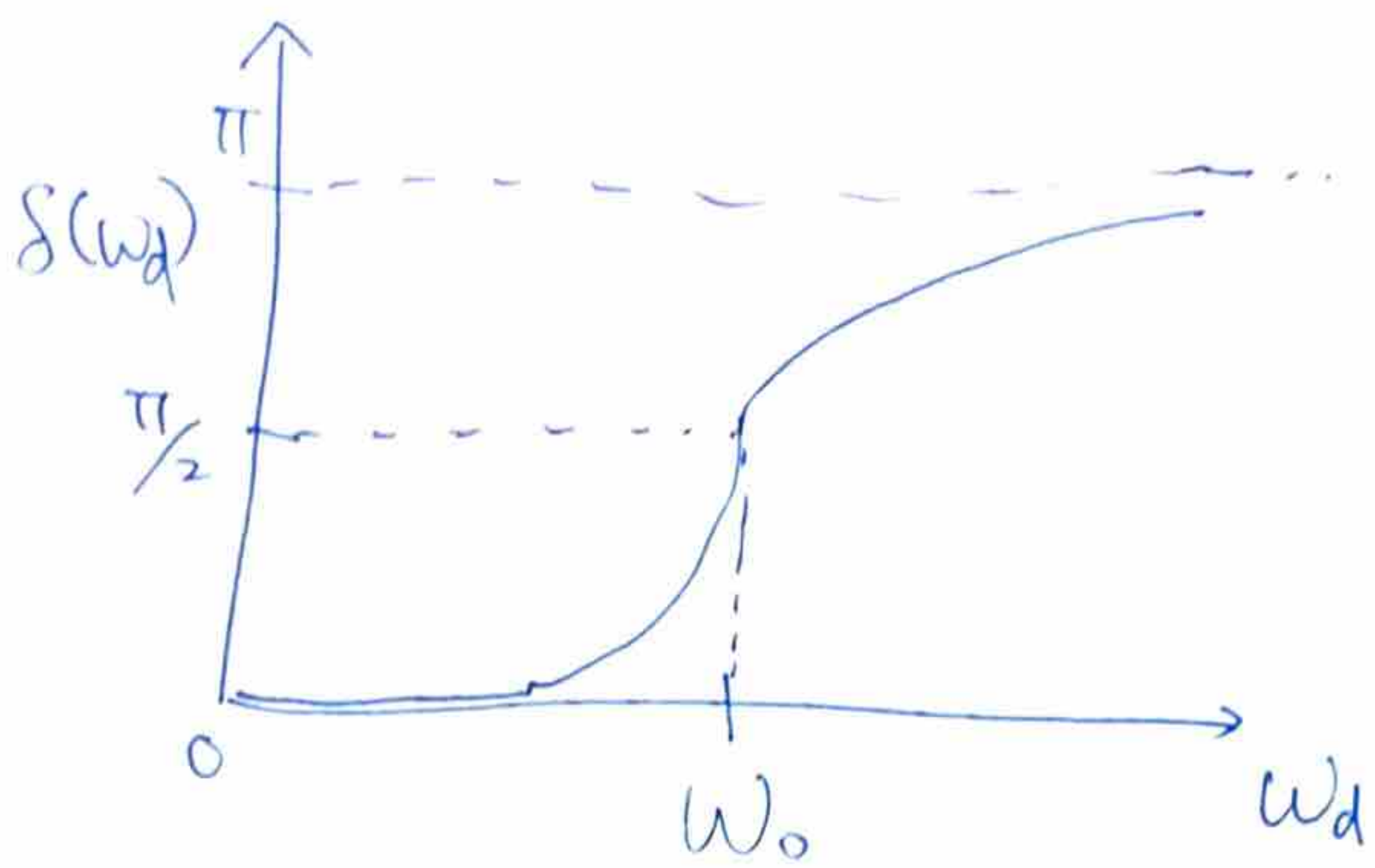
②

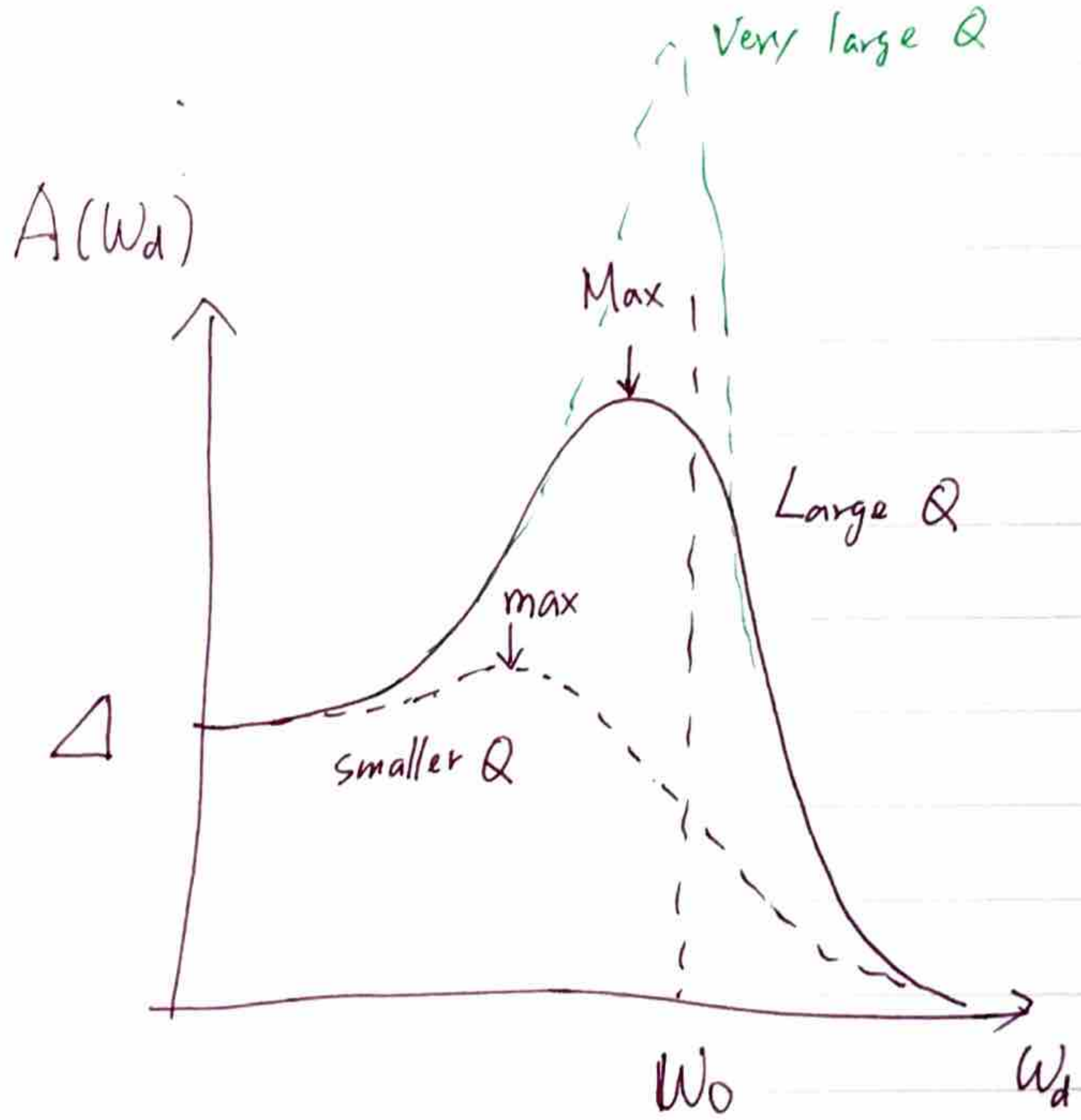
$\tan\delta = 0 \quad \to \quad \delta = 0$

(2) $W_d \to \infty$

① $A(W_d) \Rightarrow 0$

② $\tan\delta \to \infty \quad \to \quad \delta = \pi$





ω_{max} is slightly smaller than ω_0

DEMO.

(1) Move extremely slowly

We can see that

$$A(\omega_d) = \Delta$$

$$\delta = 0 \qquad \text{(No delay)}$$

(2) Move extremely fast

$A(\omega_d)$ very small

$$\delta = \pi$$

(3) $\omega_d \approx \omega_0$

resonance. Small Δ can produce large A

$$A(\omega_0) = \frac{f_0}{\omega_0 \Gamma} = \frac{\omega_0^2 \Delta}{\omega_0 \Gamma} = \frac{\omega_0}{\Gamma}\Delta$$

$$= Q\Delta$$

if $Q \equiv \dfrac{\omega_0}{\Gamma}$ is large

$\Rightarrow$ Large A

* Demo Driven Torsional Balance Oxillator

* Demo. Driven mechanical oscillator

* Air condition

* Washing Machine start to walk around

* Z bosons (slide 9)

* Lady breaks the glass by singing

Demo: Break the glass.

15 minites