8.03 **Practice** Final Exam **2** Solution

Problem 1

1.1 (3 points)

The potential energy

$$U(x) = A(1 - \cos(\alpha x)) \tag{1}$$

The minimum is at the points

$$x = 2n\pi \ , \ n \in \mathbb{Z} \tag{2}$$

For example we consider $x = 0$,

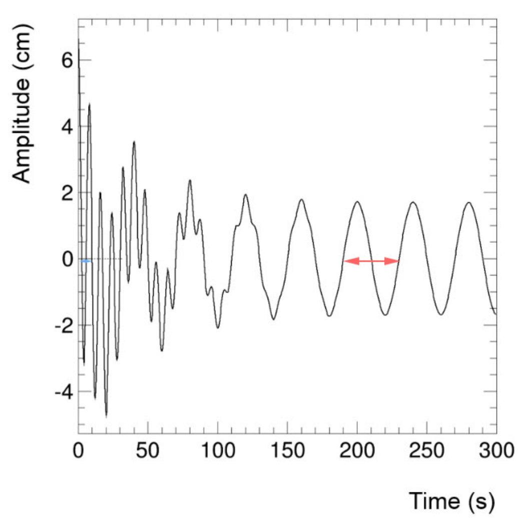
$$\left.\frac{d^2U}{dx^2}\right|_0 = A\alpha^2 \cos(\alpha x)|_0 = A\alpha^2 \tag{3}$$

This is equivalent to the parameter k in the simple harmonic motion of a spring: $U = \frac{1}{2}kx^2$. Hence we get the period

$$T = 2\pi\frac{1}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{A\alpha^2}} \tag{4}$$

1.2 (3 points)

The transient behaviour of a driven oscillation system is described by the sum of the steady state motion and the undriven decaying motion under damping.



The period of steady state motion is indicated by the red line, which is much longer than the period of undriven motion (the blue line). Also the overall shape of the transient motion is in the form of underdamped decaying motion. Hence the answer is b,f.

1.3 (3 points)

The relation between bandwidth Δf and time resolution Δt is

$$\Delta t \Delta f \sim 1 \tag{5}$$

Hence the band width

$$\Delta f \sim 10^9 Hz \tag{6}$$

1.4 (3 points)

No matter how weak the electron source is, there's always interference pattern on the screen after a long time. The answer is a,d.

1.5 (3 points)

The 2d wave equation is

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) \tag{7}$$

The lowest mode that respects the given boundary condition is

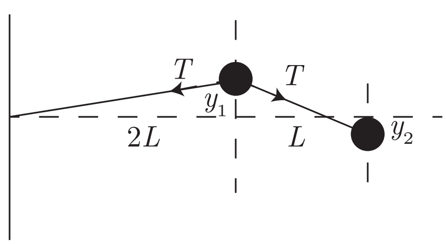
$$\psi_{1,1}(x, y, t) \sim \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{L}\right) \cos(\omega t) \tag{8}$$

Insert it into the wave equation, we get

$$\omega = \sqrt{2} \frac{v\pi}{L} \tag{9}$$

Problem 2(15 points)

a) (5 points)



The equation of motion for m_1 is

$$m\ddot{y}_1 = -\frac{T}{2L} y_1 - \frac{T}{L}(y_1 - y_2) = -\frac{3T}{2L} y_1 + \frac{T}{L} y_2 \tag{10}$$

Similarly the equation of motion for m_1 is

$$m\ddot{y}_2 = -\frac{T}{2L} y_2 - \frac{T}{L}(y_2 - y_1) = -\frac{3T}{2L} y_2 + \frac{T}{L} y_1 \tag{11}$$

Hence the matrix elements for K is

$$K_{11} = K_{22} = \frac{3T}{2L}\ ,\ K_{12} = K_{21} = -\frac{T}{L} \tag{12}$$

And the matrix $M^{-1}K$ is

$$M^{-1}K = \begin{pmatrix} \frac{3T}{2mL} & -\frac{T}{mL} \\ -\frac{T}{mL} & \frac{3T}{2mL} \end{pmatrix} \tag{13}$$

b) (5 points)

The system is symmetric under the horizontal reflection about the center point, or the interchange $y_1 \leftrightarrow y_2$. The only solutions that have this symmetry are $y_1(t) = y_2(t)$ and $y_1(t) = -y_2(t)$. They give rise to eigenvectors

$$V_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \ V_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix} . \tag{14}$$

Plug those eigenvectors into the matrix equation

$$M^{-1}KV = \omega^2 V \tag{15}$$

We get the corresponding angular frequencies for $V_\pm$:

$$\omega_+ = \sqrt{\frac{T}{2mL}} \ , \ \omega_- = \sqrt{\frac{5T}{2mL}} \tag{16}$$

c) (5 points)

The most general motion in terms of those normal modes is:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_+ t + \varphi_+) + A_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_- t + \varphi_-) \tag{17}$$

where A_+ and A_- are two coefficients fixed by the initial conditions.

From the information of initial positions and velocities, we have equations:

$$\begin{pmatrix} 0 \\ A \end{pmatrix} = A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\varphi_+ + A_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\varphi_- \tag{18}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = -A_+ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin\varphi_+ - A_- \begin{pmatrix} 1 \\ -1 \end{pmatrix} \sin\varphi_- \tag{19}$$

From the second equation we get $\varphi_+ = \varphi_- = 0$, and from the first equation we can solve

$$A_+ = \frac{A}{2} \ , \ A_- = -\frac{A}{2} \tag{20}$$

The motion for m_1 is then

$$y_1(t) = \frac{A}{2}(\cos(\omega_+ t) - \cos(\omega_- t)) \tag{21}$$

Problem 3 (20 points)

a) (5 points)

At $x = 0$, the pressure is the same as the atmosphere pressure P_0, the pressure disturbance $P(0,t) = 0$. At $x = L$, the displacement of air molecules is zero, then

$$\frac{\partial P}{\partial x}(L,t) = 0 \tag{22}$$

b) (5 points)

From the boundary condition at $x = 0$ we know the normal modes are in form of

$$P(x,t) \sim \sin(kx) \tag{23}$$

Then from the boundary condition at $x = L$:

$$k \cos(kL) = 0, \tag{24}$$

we derive

$$k = \left(n + \frac{1}{2}\right)\frac{\pi}{L}\ ,\ n = 0, 1, 2, \ldots \tag{25}$$

The Fourier expansion of $P(x,t)$ is then

$$P(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\left(n + \frac{1}{2}\right)\frac{\pi}{L}x\right)\cos\left(\left(n + \frac{1}{2}\right)\frac{v\pi}{L}t\right) \tag{26}$$

(The phases in each term are all zero because the initial “velocity” $\frac{\partial P}{\partial t} = 0$)

c) (5 points)

The Fourier coefficients can be solved by evaluating the integration

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\left(n + \frac{1}{2}\right)\frac{\pi}{L}x\right)dx \tag{27}$$

$f(x)$ is the initial shape given in the problem.

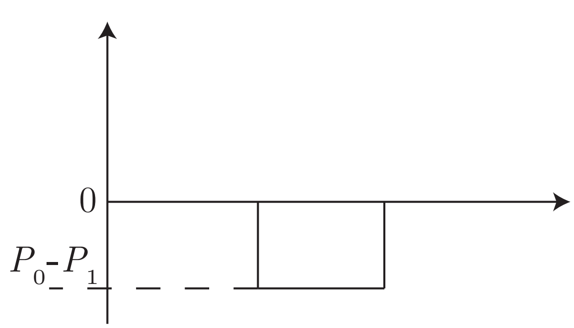
$$\begin{aligned} A_n &= \frac{2(P_1 - P_0)}{L}\int_{L/3}^{2L/3} \sin\left(\left(n + \frac{1}{2}\right)\frac{\pi}{L}x\right)dx \\ &= -\frac{2(P_1 - P_0)}{\left(n + \frac{1}{2}\right)\pi}[\cos\left(\left(n + \frac{1}{2}\right)\frac{2\pi}{3}\right) - \cos\left(\left(n + \frac{1}{2}\right)\frac{\pi}{3}\right)] \end{aligned} \tag{28}$$

d) (5 points)

At time $t = \frac{2L}{v}$, the arguments

$$\left(n + \frac{1}{2}\right)\frac{v\pi}{L}t = 2\pi n + \pi \tag{29}$$

Hence all the $\cos\left(\left(n + \frac{1}{2}\right)\frac{v\pi}{L}t\right)$ in the expansion of $P(x,t)$ equal to -1. The configuration (pressure disturbance) is then



Problem 4 (15 points)

a) (3 points)

The average acceleration

$$\langle \vec{a} \rangle = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} \tag{30}$$

where $\vec{v}_1 = \omega \hat{x}$, $\vec{v}_2 = \omega \cos \Delta\alpha \cdot \hat{x} - \omega \sin \Delta\alpha \cdot \hat{y}$ are the initial and final velocity. Since $\Delta\alpha \ll 1$, $\cos \Delta\alpha - 1$ is a second order infinitesimal quantity, which can be ignored comparing to $\sin \Delta\alpha$. Then

$$\vec{v}_2 - \vec{v}_1 = -\omega \Delta\alpha \cdot \hat{y} \tag{31}$$

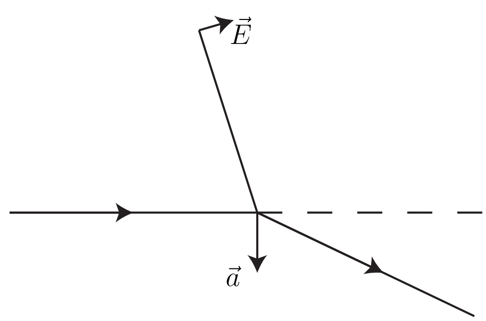
$$\langle \vec{a} \rangle = -\frac{\omega \Delta\alpha}{\Delta t} \cdot \hat{y} \tag{32}$$

b) (3 points)

The electric field generated by this acceleration is

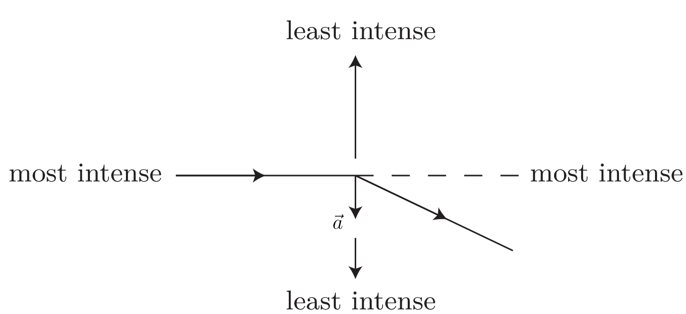
$$\vec{E}(\vec{r}, t) = -\frac{q \vec{a}_\perp (t - |r|/c)}{4\pi\epsilon_0 r c^2} \tag{33}$$

where $\vec{a}_\perp(t - |r|/c)$ is the acceleration projected to the direction transverse to $\vec{r}$. Hence the direction of electric field is shown below:



c) d) (2 points for each)

The direction perpendicular to the direction of acceleration has the most intense radiation. The direction along or opposite the direction of acceleration has least radiation (no radiation):



e) (2 points)

From (33) we know that the electric field scales like $\vec{E} \sim \frac{1}{r}$. Hence the amplitude at P_2 will be $\frac{1}{2}$ of the amplitude at P_1.

f) (3 points)

The total radiation power is

$$P(t) = \frac{q^2 a^2 (t - |r|/c)}{6\pi\epsilon_0 c^3} \tag{34}$$

Hence the total energy radiated by charge is

$$E = P\Delta t = \Delta t \cdot \frac{q^2\omega^2\Delta\alpha^2}{6\pi\epsilon_0 c^3 (\Delta t)^2} = \frac{q^2\omega^2\Delta\alpha^2}{6\pi\epsilon_0 c^3 \Delta t} \tag{35}$$

Problem 5(15 points)

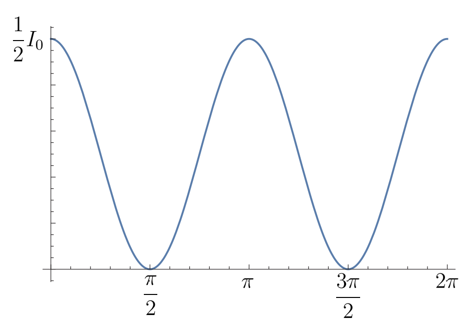
a) (5 points)

For unpolarized light, the intensity after passing through a linear polarizer is $I_A = \frac{1}{2}I_0$. Its polarization after passing through A is along $\hat{x}$ direction.

b) (5 points)

After the light passed through B, the polarization vector is projected to the easy direction of B, hence the final intensity is

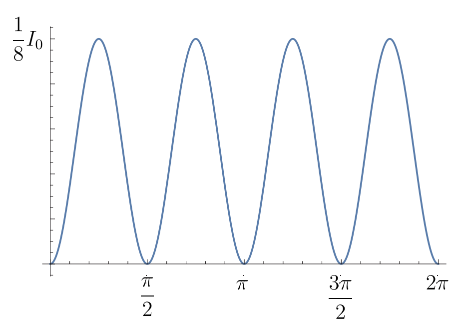
$$I_B = I_A \cos^2\theta = \frac{1}{2}I_0 \cos^2\theta \tag{36}$$



c) (5 points)

After the light passed through C, the polarization vector is projected from the easy direction of B to the easy direction of C ($\hat{y}$-direction), hence the final intensity is

$$I_C = I_B \sin^2\theta = \frac{1}{2}I_0 \cos^2\theta \sin^2\theta = \frac{1}{8}I_0 \sin^2(2\theta) \tag{37}$$



Problem 6(20 points)

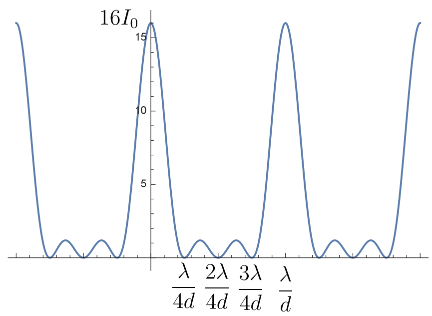
a) (4 points)

Since the slits are very narrow, the intensity is just the 4-slits interference intensity:

$$I = I_0 \left(\frac{\sin^2(2\delta)}{\sin^2 \frac{\delta}{2}} \right) \quad , \ \delta = 2\pi \frac{d}{\lambda} \sin\psi \tag{38}$$

b) (4 points)

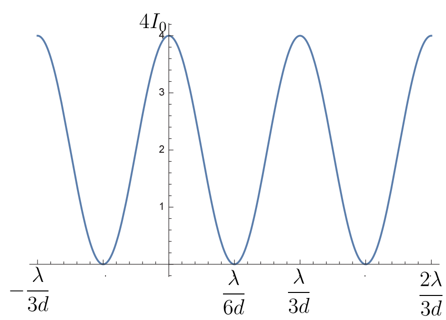
The principal maximas are at $\delta = 2\pi n$ or $\sin\psi = n\frac{\lambda}{d}$. The minimas are at $\delta = \frac{m\pi}{2}$, where $4 \nmid m$.



c) (4 points)

When the two middle slits are closed, the system is just a two slits interference with slit distance $3d$. Hence

$$I = I_0 \left(\frac{\sin^2 \delta'}{\sin^2 \frac{\delta'}{2}} \right) \quad , \ \delta' = 6\pi \frac{d}{\lambda} \sin\psi \tag{39}$$



d) (4 points)

The new maximas are at

$$\sin\psi = n\frac{\lambda}{3d} \ (n \in \mathbb{Z}) \tag{40}$$

and the new minimas are at

$$\sin\psi = \left(n + \frac{1}{2}\right) \frac{\lambda}{3d} \ (n \in \mathbb{Z}) \tag{41}$$

The principal maximas

$$\sin\psi = n\frac{\lambda}{d} \ (n \in \mathbb{Z}) \tag{42}$$

are at same positions.

The intensity of principal maximas decreased from $16I_0$ to $4I_0$.

e) (4 points)

When the width of a single slit cannot be ignored, the intensity

$$I = I_0 \left(\frac{\sin^2(2\delta)}{\sin^2 \frac{\delta}{2}} \right) \frac{\sin^2 \beta}{\beta^2} \ , \ \beta = \pi \frac{D}{\lambda} \sin \psi \tag{43}$$

The zero points of diffraction factor $\frac{\sin^2 \beta}{\beta^2}$ are at

$$\sin \psi = n \frac{\lambda}{D} \ , \ n = 1, 2, 3, \ldots \tag{44}$$

The condition which principal maxima of the interference pattern overlaps with the diffraction zero point is then

$$n \frac{\lambda}{D} = \frac{m\lambda}{d} = \frac{m\lambda}{5D} \tag{45}$$

So the lowest interference order for this to happen is $m = 5$.

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