8.022 (E&M) – Lecture 20

Topics:

- Electromagnetic plane waves and their properties
- Polarization of EM waves
  - Polaroids and linear and circular polarization

Last time

- Completed Maxwell's equations
  - Displacement currents
    - Kirchoff's laws are legitimate!

$$\begin{cases} \vec{\nabla} \bullet \vec{E} = 4\pi\rho \\ \vec{\nabla} \bullet \vec{B} = 0 \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \boxed{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}} \end{cases}$$

- Solved Maxwell's equations in vacuum
  - Derived wave equation for EM waves
  - They travel at speed of light: light is EM wave!
  - Started studing properties of the general solution f(x-vt)

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

- Today we will complete the study of these properties…

Plane waves

- Fourier Theorem:
  - Any periodic function can be expressed as a linear combination of sin and cos functions

    → sin and cos are the building blocks of all waves!
- Plane waves in the most general form:
  - $\vec{E} = \vec{E}_0 \sin(\vec{k}\bullet\vec{r} - \omega t) = \vec{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)$
  - $\vec{B} = \vec{B}_0 \sin(\vec{k}\bullet\vec{r} - \omega t) = \vec{B}_0 \sin(k_x x + k_y y + k_z z - \omega t)$

  where:

  $\vec{k}$ = wavevector; $|\vec{k}|$ = wavenumber; $\hat{k}$ = propagation direction

Plane waves vs f(x-ct)

- We proved that $f(x \pm ct)$ satisfies the wave equation
- How to connect $(x \pm ct)$ to the the argument of plane waves $(\vec{k}\bullet\vec{r} \pm \omega t)$?
  - From 1D to 3D:

    $$f(x \pm ct) \Rightarrow f(\vec{r} \pm c\hat{k}t)$$
  - Relation between k, ω and c:

    $$\vec{k}\bullet\vec{r} \pm \omega t = \vec{k}\bullet(\vec{r} \pm \frac{\omega}{k}\hat{k}t) = \vec{k}\bullet(\vec{r} \pm c\hat{k}t)$$

    → $\omega = ck$

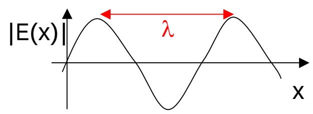
More on k and ω

- Choose a system of coordinates so that our wave vector k is oriented // to x axis: plane wave solution for E is

$$\vec{E} = \vec{E}_0 \sin(k_x x - \omega t)$$

- Let's consider only the spatial variation of the wave (e.g. t=0):

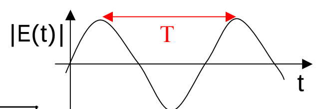
$$\vec{E} = \vec{E}_0 \sin(k_x x)$$



  - λ = wavelength

- Let's now consider the time variation of the wave (e.g. x=0):

$$\vec{E} = \vec{E}_0 \sin(\omega t)$$



- Relations between variables:

$$\boxed{\omega = \frac{2\pi}{T} = 2\pi\nu} \quad \boxed{\omega = ck} \quad \boxed{\lambda\nu = c}$$

Do plane wave satisfy Maxwell's equations?

- EM waves are a consequence of Maxwell's equations in the sense that we used the 4 Maxwell's Equations to derive the wave equations for E and B:

$$\begin{cases} \vec{\nabla}^2 \vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2} \\ \vec{\nabla}^2 \vec{B} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{B}}{\partial t^2} \end{cases} \Rightarrow \begin{cases} \vec{E} = \vec{E}_0 \sin(\vec{k}\bullet\vec{r} - \omega t) \\ \vec{B} = \vec{B}_0 \sin(\vec{k}\bullet\vec{r} - \omega t) \end{cases}$$

- Does the solution of the EM wave equation satisfy all Maxwell's Equations?
  - Not necessarily! Let's start with Gauss's law: $\vec{\nabla}\bullet\vec{E} = 0$

$$\nabla\bullet\vec{E} = \nabla\bullet\left[\vec{E}_0 \sin(k_x x + k_y y + k_z z - \omega t)\right] =$$

$$(\vec{E}_{0x}k_x + \vec{E}_{0y}k_y + \vec{E}_{0z}k_z)\cos(k_x x + k_y y + k_z z - \omega t) = \vec{k}\bullet\vec{E}_0 \cos(\vec{k}\bullet\vec{r} - \omega t)$$

$\Rightarrow \nabla\bullet\vec{E} = 0$ when $\vec{k}\bullet\vec{E}_0 = 0$ $\Rightarrow$ $\vec{E}$ is $\perp$ to wave's direction of propagation

More constraints on plane waves

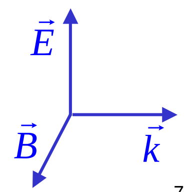
- Constraints following from $\vec{\nabla}\bullet\vec{B} = 0$

  $\nabla\bullet\vec{B} = \vec{k}\bullet\vec{B}_0 \cos(\vec{k}\bullet\vec{r} - \omega t) = 0 \Rightarrow \vec{k}\bullet\vec{B} = 0$

  $\Rightarrow \vec{B}$ is $\perp$ to direction of propagation

- Constraints following from $\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$
  - Time derivative does not change direction of B → $\vec{E} \perp \vec{B}$
- Same conclusion follows from: $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

Conclusion: $\vec{k} \perp \vec{E} \perp \vec{B} \perp \vec{k}$



More constraints on plane waves

- Let's now calculate: $\vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$

  $$\frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{\omega}{c}\vec{E}_0 \sin(\vec{k}\bullet\vec{r} - \omega t) = k\vec{E}_0 \sin(\vec{k}\bullet\vec{r} - \omega t)$$

  $$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times\vec{B}_0 \cos(k_x x + k_y y + k_z z - \omega t)$$

- Using $\vec{\nabla}\times(\vec{v}s) = \vec{\nabla}\times\vec{v} + \vec{\nabla}s\times\vec{v}$ and the fact that B_0=const

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times\left[\vec{B}_0 \cos(k_x x + k_y y + k_z z - \omega t)\right] =$$

$$= \left(\vec{\nabla}\times\vec{B}_0\right)\cos(\vec{k}\bullet\vec{r} - \omega t) + \vec{\nabla}\cos(k_x x + k_y y + k_z z - \omega t)\times\vec{B}_0 =$$

$$= -\left(k_x\hat{x} + k_y\hat{y} + k_z\hat{z}\right)\sin(k_x x + k_y y + k_z z - \omega t)\times\vec{B}_0 =$$

$$= -\left(\vec{k}\times\vec{B}_0\right)\sin(\vec{k}\bullet\vec{r} - \omega t)$$

More constraints on plane waves

- From $\vec{\nabla} \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$ it follows that

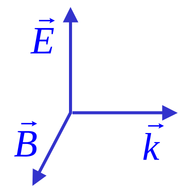
$$-\left(\vec{k} \times \vec{B}_0\right)\sin(\vec{k} \bullet \vec{r} - \omega t) = k\vec{E}_0 \sin(\vec{k} \bullet \vec{r} - \omega t)$$

$$\Rightarrow \boxed{\vec{E}_0 = -\hat{k} \times \vec{B}_0} \text{ or } \vec{k}, \vec{E} \text{ and } \vec{B} \text{ are right handed ortogonal vectors}$$

- Important consequences:
  - In cgs, E and B have the same magnitude

    $$\left|\vec{E}_0\right| = \left|-\hat{k} \times \vec{B}_0\right| \Rightarrow \boxed{E_0 = B_0}$$

  - $\vec{E}_0 \times \vec{B}_0$ is parallel to the propagation of wave
    - $\vec{E}_0 = -\hat{k} \times \vec{B}_0 \Rightarrow \vec{E}_0 \times \vec{B}_0 = \left|\vec{E}_0\right|^2 \hat{k}$
    - NB: E x B has an important physical meaning that we will soon see

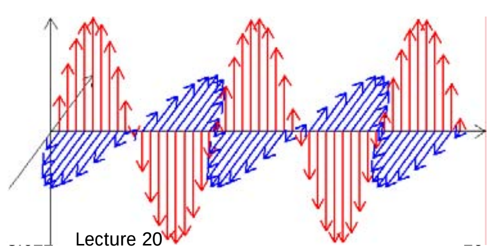


Polarization of EM waves

- Did we use all of our freedom in choosing the waves?
  - No, we can still choose the so called "polarization state"
- Linear polarization:
  - Consider a plane wave propagating in the x direction
  - Choose the coordinate system so that at t=0 $\vec{E} \,//\, \hat{y}$ and $\vec{B} \,//\, \hat{z}$
  - If the directions of E_0 and B_0 are constant in time, the wave is "linearly polarized"

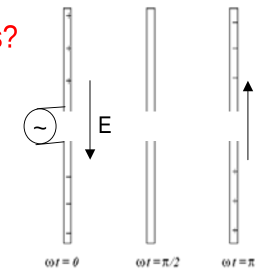
$$\begin{cases} \vec{E} = E_0 \cos(kx - \omega t)\hat{y} \\ \vec{B} = B_0 \cos(kx - \omega t)\hat{z} \end{cases}$$

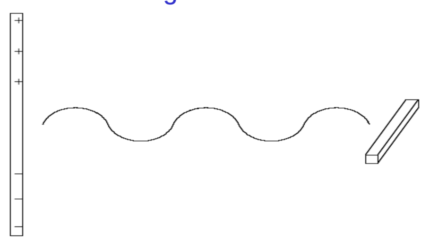
NB: direction of polarization = direction of electric field



Linear Polarization of EM waves

- How to produce linearly polarized waves?
  - Oscillating charge distribution in a conductor
    - Broadcasting antenna
  - How to produce such charge?
    - Long conductor driven by oscillating current
- How do we receive the signal?
  - Receiving antenna





When receiver is perpendicular to broadcasting antenna:
no reception because there is not enough room for charges to oscillate

Demo K1

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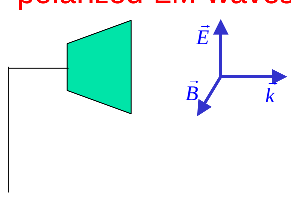
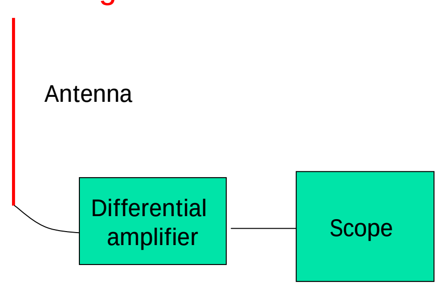
Demo K1: E of microwaves

- A microwave generator produces a signal of 10.5 GHz polarized EM waves





- How should we orient antenna to detect a signal on the scope?
  - Antenna // E will detect signal
  - Antenna perpendicular to E: no signal

Polaroids

- Sheet of plastic embedded with organic molecules extended in one direction
  - They can carry current in that particular direction: behave like antennas!
- When linearly polarized light hits the polaroid:
  - If E is aligned with orientation of molecules:
    - Charges move → current is generated → plastic heats: light stopped
  - If E is perpendicular to orientation of molecules ("preferred direction"):
    - Charges will not be able to move in that direction: light goes through

Conclusion:

- Polaroids are transparent to light polarized // to their preferred direction and opaque to light polarized in the direction perpendicular to their preferred direction

Polaroids and polarization direction

- What happens when the light is polarized in a direction in between the preferred direction and its perpendicular?
  - Example: light polarized along x axis; polaroid oriented at θ angle

$$\vec{E} = E_0 \cos(kz - \omega t)\hat{x}$$

$$\hat{p} = \hat{x}\cos\theta + \hat{y}\sin\theta$$

- Light will go through partially
  - Since E has a component // to preferred direction of polaroid
  - E coming out is overlap between incoming E and polaroid's orientation

$$\left|\vec{E}_{out}\right| = \vec{E}\bullet\hat{p} = E_0\cos(kz-\omega t)\left(\hat{x}\cos\theta + \hat{y}\sin\theta\right)\bullet\hat{x} = E_0\cos\theta\cos(kz-\omega t)$$

$$\vec{E}_{out} = \left|\vec{E}\bullet\hat{p}\right|\hat{p} \text{ (parallel to polaroid's orientation)}$$

Conclusion:

Polaroids reduce the amplitude of linearly polarized light by cosθ (angle between E and polaroid's orientation) and rotates the orientation of E by θ

Polarization of random light

- Light from a bulb, sunlight, etc is not polarized
  - Superposition of many plane waves, each with its own polarization

$$\vec{E}_{random} = \sum_i E_0(\hat{x}\cos\theta_i + \hat{y}\sin\theta_i)\cos(kz-\omega t)$$

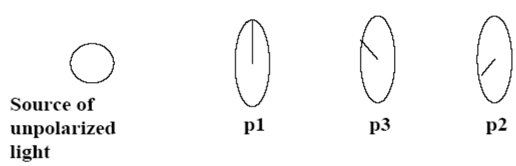
- When light passes through a polaroid becomes linearly polarized
  - If polaroid is oriented // x axis:

$$\vec{E}_{out} = \sum_i E_0(\hat{x}\cos\theta_i)\cos(kz-\omega t) = E_0\hat{x}\cos(kz-\omega t)\sum_i \cos\theta_i$$

- Conclusion:
  - Polaroids can be used to produce linearly polarized light
  - The intensity of the light will be reduced

Demo: 3 vs 2 polaroids

- 2 polaroids with orthogonal preferred direction will block light
  - First polaroid (P1) polarizes light in the direction x (for example)
  - Second polaroid (P2)oriented in the y direction, but E is now just //x

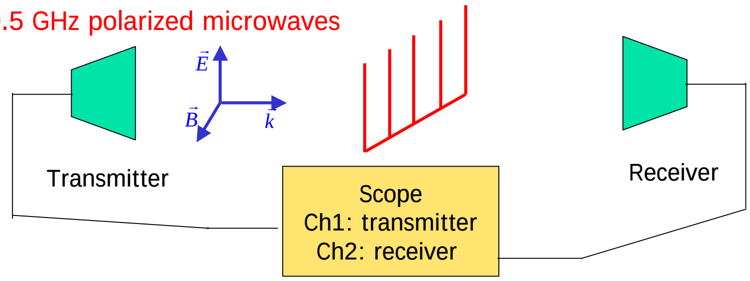


DEMO T1

- Now place a third polaroid P3 in between p1 and p2 (at 45 degrees)
  - P1 will polarize light //x
  - P3 will select only component // to its preferred direction and rotate direction of polarization by 45 deg. $E_0' = E_0 \cos 45^o$
  - P2 will select component y direction that now is not 0 anymore. Intensity further reduced, but not 0! $E_0' = E_0 (\cos 45^o)^2 = E_0/2$

Polarization of microwaves (K3)

- 10.5 GHz polarized microwaves



- Rotate the receiver to find the direction of polarization of signal
- Now introduce a conductive "comb" in between transmitter and receiver
  - When teeth of comb are // E: signal is blocked
  - When they are perpendicular to E: signal can go through

  → Exactly the same behavior of Polaroid for light!

Circular polarization

- Consider a wave with the following form:

$$\vec{E} = E_0 \hat{x} \sin(kz - \omega t) + E_0 \hat{y} \cos(kz - \omega t)$$

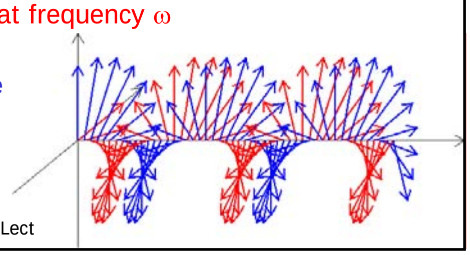
$$\vec{B} = B_0 \hat{y} \sin(kz - \omega t) - B_0 \hat{x} \cos(kz - \omega t)$$

- What is it?
  - Easier to understand if we look at z=0

$$\vec{E} = -E_0 \hat{x} \sin(\omega t) + E_0 \hat{y} \cos(\omega t)$$

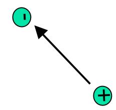
$$\vec{B} = -B_0 \hat{y} \sin(\omega t) - B_0 \hat{x} \cos(\omega t)$$

- Electric and magnetic fields rotate at frequency ω
  - Circular polarization because E and B vectors describe circles over time

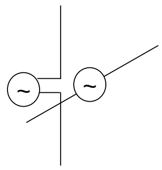


Circular polarization (2)

- How to produce it?
  - Rotating dipole



  - 2 antennas at 90 deg driven by currents off by 90 deg

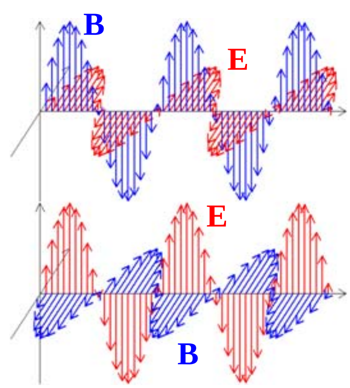


- NB: circular polarization does exist in nature
  - Example: circular polarization filters used in photography

Elliptical Polarization

- For a given **k**, there are 2 independent solutions for the plane waves, e.g. 2 possible directions of **E**

$$\vec{E}_1 = E_0 \hat{x} \cos(kz - \omega t + \phi_1)$$
$$\vec{E}_2 = E_0 \hat{y} \cos(kz - \omega t + \phi_2)$$



- All other solutions are just linear combinations of these
  - $\phi_1=\phi_2$: linear polarization
  - $\phi_1=\phi_2+90^o$: linear polarization
  - All the rest: elliptical polarization

Summary and outlook

- Today:
  - Electromagnetic plane waves
    - Constraints on E, B and k following from Maxwell's equations
      - E, B and k are always perpendicular to each other
      - Amplitude of E and B are the same in cgs
  - Polarization of EM waves
    - Polaroids and linear and circular polarization
- Next Tuesday:
  - Energy and momentum carried by EM waves
    - Poynting vector
  - Transmission lines

8.022 subject evaluation

- Fast:
  - 5-10 minutes of your time
- Important:
  - Your chance to make comments about the class
- You can be honest!
  - We will not look at the forms until after grades are registered
  - A volunteer will collect the results and will bring them to the PEO(?)
- NB:
  - Fill in both sides of the form; side 1 will be read by computer
  - Staff: in addition to lecturer and recitation instructor you can fill in the evaluation for ONE tutor:
    - Michael Shaw OR Min Liang Zhao

Reward:
how to "hack" a radio station