8.022 (E&M) – Lecture 19

Topics:

- The missing term in Maxwell's equation
  - Displacement current: what it is, why it's useful
- The complete Maxwell's equations
  - And their solution in vacuum: EM waves

Maxwell's equations so far

$$\begin{cases} \vec{\nabla} \bullet \vec{E} = 4\pi\rho & \leftarrow \text{Gauss's law: relates E and charge density } (\rho) \\ \vec{\nabla} \bullet \vec{B} = 0 & \leftarrow \text{Magnetic field lines are always closed!} \\ \vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} & \leftarrow \text{Faraday's law: change in B flux creates e.m.f. (E)} \\ \vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} & \leftarrow \text{Ampere's law: relates B and its sources (J)} \end{cases}$$

Is this set of equations completely consistent?

Not quite...

Maxwell's equations so far (2)

$$\begin{cases} \vec{\nabla}\bullet\vec{E} = 4\pi\rho \\ \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \\ \boxed{\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}} \end{cases}$$

- Is this set of equations consistent? Not quite...
  - Take the divergence of Ampere's law
    - $\vec{\nabla}\bullet\left(\frac{4\pi}{c}\vec{J}\right) = \frac{4\pi}{c}\vec{\nabla}\bullet\vec{J} = -\frac{4\pi}{c}\frac{\partial\rho}{\partial t}$ (using continuity equation)
    - $\vec{\nabla}\bullet\vec{\nabla}\times\vec{B} = 0 \quad (\vec{\nabla}\bullet\vec{\nabla}\times\vec{v}$ is ALWAYS 0!)

Ampere's law works only when $d\rho/dt=0$ which works in most cases but not always: Ampere's law is incomplete!

Fixing the inconsistency

- Since $\nabla\bullet\nabla\times\vec{V}\equiv 0$ we need to add some term to the right hand side to that its divergence will be identically 0
- Generalized Ampere's law: $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \vec{F}$
- What is F? We know that its divergence must be =0:

$$\vec{\nabla}\bullet\left(\frac{4\pi}{c}\vec{J}+\vec{F}\right) = 0 \Rightarrow \vec{\nabla}\bullet(c\vec{F}) = -4\pi\vec{\nabla}\bullet\vec{J} = 4\pi\frac{\partial\rho}{\partial t}$$

$$\Rightarrow \boxed{\vec{\nabla}\bullet(c\vec{F}) = 4\pi\frac{\partial\rho}{\partial t}} \text{ Similar to Gauss's law!}$$

- Take time derivative of Gauss's law:

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{E}\right) = 4\pi\frac{\partial\rho}{\partial t} \Rightarrow \frac{\partial}{\partial t}\left(\vec{\nabla}\bullet\vec{E}\right) = \vec{\nabla}\bullet\left(\frac{\partial\vec{E}}{\partial t}\right) = \vec{\nabla}\bullet(c\vec{F}) \text{ time and space derivatives commute}$$

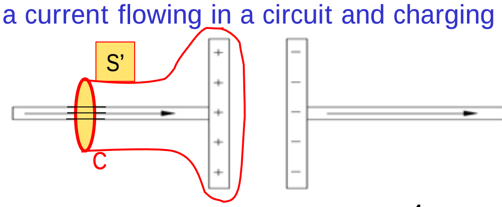
$$\Rightarrow \boxed{\vec{F} = \frac{1}{c}\frac{\partial\vec{E}}{\partial t}}$$

Displacement currents

- Generalized Ampere's equation $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$
- This can also be written as: $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}(\vec{J} + \vec{J}_d)$
  - With J_d = displacement current (density): $\vec{J}_d = \frac{1}{4\pi}\frac{\partial\vec{E}}{\partial t}$
- What is the J_d?
  - Not a real current: does not describe charges flowing through some region
  - But it acts like a real current: whenever we have changing E field, we can treat its effect as if due to as a real current J_d

What is a displacement current?

- Consider a current flowing in a circuit and charging a capacitor C



- Standard integral Ampere's law: $\oint_C \vec{B}\bullet d\vec{l} = \frac{4\pi}{c}I_{encl} = \frac{4\pi}{c}\int_S \vec{J}\bullet d\vec{a}$
  - Let's choose the path C and the surface S as in the drawing above:
    - It all makes sense!
  - Now choose the same path C but the surface S' (ok by Stokes...)
    - No standard current J through the surface (no charge crosses C!)
    - But there is a flux of displacement current J_d through the plates!

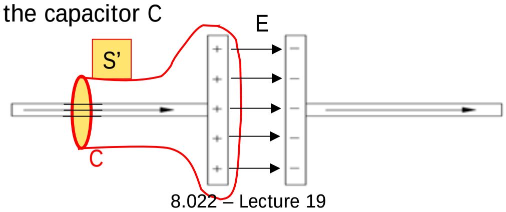
What is a displacement current? (2)

- We can use the generalized Ampere's Law:

$$\oint_C \vec{B} \bullet d\vec{l} = \frac{4\pi}{c}\left(I_{encl} + I_d\right)$$

$$\text{with } I_d = \int_{S'} \vec{J}_d \bullet d\vec{a} = \frac{1}{4\pi}\int_{S'} \frac{\partial \vec{E}}{\partial t} \bullet d\vec{a} = \frac{1}{4\pi}\frac{\partial}{\partial t}\int_{S'} \vec{E} \bullet d\vec{a} = \frac{1}{4\pi}\frac{\partial \Phi_{\vec{E}}}{\partial t}$$

- The displacement current is related to the change over time of the flux of the electric field.
  - In the example above, the electric field E is the one produced in between the plates of the capacitor C



What is a displacement current? (3)

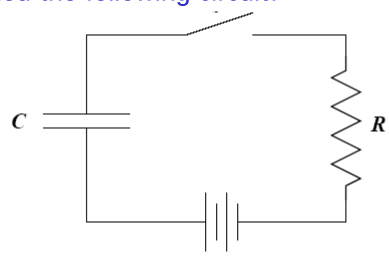
- The electric field E:
  - Points in the same direction of the current (+x)
  - At a given instant in time: $\vec{E} = \frac{4\pi Q}{A}\hat{x}$
- The flux of E will then be: $\Phi_{\vec{E}} = 4\pi Q$ (yes, Gauss's law!)
- The rate of the change if this flux is: $\frac{\partial \Phi_{\vec{E}}}{\partial t} = 4\pi\frac{\partial Q}{\partial t} = 4\pi I$
  - Where I is the current that is charging the capacitor
- Comparing this with results in the previous page:

$$I_d = \int_{S'} \vec{J} \bullet d\vec{a} = \int_S \vec{J}_d \bullet d\vec{a} = I$$

→ generalized Ampere's Law is valid no matter what surface we use

The importance of displacement currents

- When we examined the following circuit:



we said the same current **I** was flowing in each circuit element.

- How is it possible? No current flows through the plates of a capacitor!
  - Displacement currents fix this inconsistency!
  - Displacement current “continues” the “real” current across the capacitors ensuring the validity of Kirchoff’s laws.

Maxwell’s equations (complete!)

cgs

$$\begin{cases} \vec{\nabla}\bullet\vec{E} = 4\pi\rho \\ \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} = \dfrac{4\pi}{c}\vec{J} + \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

SI

$$\begin{cases} \vec{\nabla}\bullet\vec{E} = \dfrac{\rho}{\varepsilon_0} \\ \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

Typo in Purcell Eq 15 ch 9

Generalized Ampere’s law

NB: when Maxwell introduced the term dE/dt in the generalized Ampere’s law, his arguments were based purely on symmetry

- Yes, he was a theorist!

Maxwell's equations: integral form

$$\begin{cases} \Phi_{\vec{E}} = \int_S \vec{E} \bullet d\vec{a} = 4\pi Q_{enc} & \text{(Gauss's law)} \\ \Phi_{\vec{B}} = 0 & \text{(Magnetic field line are closed)} \\ emf = \oint_C \vec{E} \bullet d\vec{l} = -\frac{1}{c}\frac{\partial \Phi_{\vec{B}}}{\partial t} & \text{(Faraday's law)} \\ \oint_C \vec{B} \bullet d\vec{l} = \frac{4\pi}{c}(\vec{I} + \vec{I}_d) & \text{(Generalized Ampere's law)} \end{cases}$$

where the currents $\vec{I}$ and $\vec{I}_d$ are defined as $\vec{I} = \int_S \vec{J} \bullet d\vec{a}$ and

$$\vec{I}_d = \frac{1}{4\pi}\frac{\partial \Phi_{\vec{E}}(S)}{\partial t}$$

3 good reasons to remember Maxwell's equations

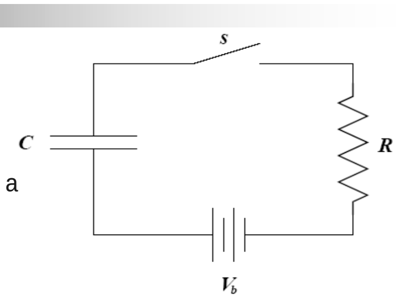
1) They compactly and beautifully summarize all the E&M we learned so far!

2) You will see them on T shirts for the rest of your life at MIT: better to get familiar with them ASAP!

3) On the first day of 8.03 next semester you will be asked to write them down on a piece of paper to check what you learned in your first semester at MIT: save your honor (and mine)

Displacement current: application

- Consider the following RC circuit:
  - As C charges up, I_d flows
  - I_d induces B inside the plates
  - Assuming cylindrical plates of radius a



- Calculate B inside the plates

1) Find $\vec{E}(t)$: $\vec{E}(t) = 4\pi\sigma = \dfrac{4\pi Q(t)}{\pi a^2}$

2) Displ. current density: $\vec{J}_d = \dfrac{1}{4\pi}\dfrac{\partial \vec{E}}{\partial t} = \dfrac{1}{4\pi}\dfrac{\partial E(t)}{\partial t} = \dfrac{1}{\pi a^2}\dfrac{\partial Q(t)}{\partial t} = \dfrac{I(t)}{\pi a^2}$

3) Remember that $I(t) = \dfrac{V_b}{R}e^{-t/RC}$

4) Magnetic field inside the plate (Ampere's law): $\oint_C \vec{B}\bullet d\vec{l} = \dfrac{4\pi}{c}\int_C \vec{J}_d \bullet d\vec{a}$

$$\Rightarrow B(r) = \frac{2rV_b}{ca^2R}e^{-t/RC}$$

Maxwell equations in vacuum

- What happens when we write Maxwell's equations in vacuum?
  - Vacuum: no sources, ρ=0 and J=0

$$\begin{cases} \vec{\nabla}\bullet\vec{E} = 4\pi\rho \\ \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} = \dfrac{4\pi}{c}\vec{J} + \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} \end{cases} \rightarrow \begin{cases} \vec{\nabla}\bullet\vec{E} = 0 \\ \vec{\nabla}\bullet\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} = \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} \end{cases}$$

- Except for a – sign, these equations are exquisitely symmetric!
- Consequence: an electric filed E varying in time will create a magnetic filed B; a B varying in time creates a E: E and B are intimately related!

Maxwell equations in vacuum: solution

- How to solve these equations?
  - Uncouple them!
    - Separate E and B in equations
- How?
  - Take the curl of equations (3) and (4)
  - Use other equations as needed

$$\begin{cases} \vec{\nabla}\bullet\vec{E} = 0 & (1) \\ \vec{\nabla}\bullet\vec{B} = 0 & (2) \\ \vec{\nabla}\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} & (3) \\ \vec{\nabla}\times\vec{B} = \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t} & (4) \end{cases}$$

- Start from (3):

Left: $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\left(\vec{\nabla}\bullet\vec{E}\right) - \vec{\nabla}^2\vec{E} = -\vec{\nabla}^2\vec{E}$ (since $\vec{\nabla}\bullet\vec{E} = 0$ in vacuum)

Right: $\vec{\nabla}\times\left(-\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t}\right) = -\dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{\nabla}\times\vec{B} = -\dfrac{1}{c^2}\dfrac{\partial^2\vec{E}}{\partial t^2}$ (using (4))

$$\Rightarrow \boxed{\vec{\nabla}^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

Maxwell equations in vacuum: solution

- Now repeat the procedure starting from $\vec{\nabla}\times\vec{B} = \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t}$ (4)

Left: $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{B}\right) = \vec{\nabla}\left(\vec{\nabla}\bullet\vec{B}\right) - \vec{\nabla}^2\vec{B} = -\vec{\nabla}^2\vec{B}$ (since $\vec{\nabla}\bullet\vec{B} = 0$ in vacuum)

Right: $\vec{\nabla}\times\left(\dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t}\right) = \dfrac{1}{c}\dfrac{\partial}{\partial t}\vec{\nabla}\times\vec{E} = -\dfrac{1}{c^2}\dfrac{\partial^2\vec{B}}{\partial t^2}$ (using (3))

$$\Rightarrow \boxed{\vec{\nabla}^2\vec{B} = \frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}}$$

- This is a special case of a known equation: the wave equation:

$$\boxed{\vec{\nabla}^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}} \quad \text{where } f = f(x \pm vt)$$

where f is <u>any</u> function that has well-behaved derivatives

NB: we are restricting ourselves to the 1D case; extension to 3D next lecture

$$\vec{\nabla}^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Solution of wave equation: prove

- Prove that $f = f(x \pm vt)$ is a solution of the wave equation
- Just calculate time and space derivatives.
  - Keep in mind that $\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
- Define $u = x \pm vt$

$$\frac{\partial f(x \pm vt)}{\partial t} = \frac{\partial f}{\partial u}\frac{\partial f}{\partial t} = \pm v \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(x \pm vt)}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial u^2}$$

$$\frac{\partial f(x \pm vt)}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u} \Rightarrow \frac{\partial^2 f(x \pm vt)}{\partial t^2} = \frac{\partial^2 f}{\partial u^2}$$

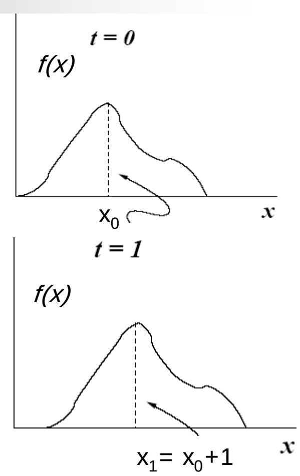
Plug the above results into the equation $\Rightarrow \frac{\partial^2 f}{\partial u^2} = \frac{1}{v^2} v^2 \frac{\partial^2 f}{\partial u^2} \Rightarrow$ identity!

- As we wanted to prove!

$$\vec{\nabla}^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Wave equation solution

- What is a function such as $f = f(x - vt)$?
  - Assume v=1 cm/s
- At time t=0:
  - Position of the max: x_0
- At time t=1 s:
  - The peak still occurs when the argument of f is x_0
  - But since the time is not 0
    → the function will be shifted in x by "vt"=1 cm
  - Position of the max: $x_1=x_0+1$

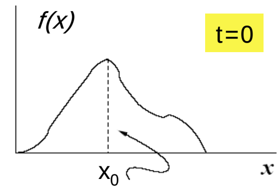


$f = f(x - vt)$ represents a wave traveling in the +x direction with velocity v

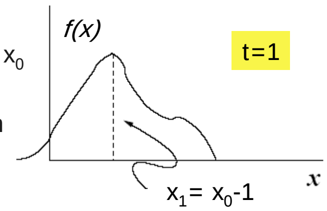
$$\vec{\nabla}^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Wave equation solution

- What is a function such as $f = f(x+vt)$?
  - Assume v=1 cm/s
- At time t=0:
  - Position of the max: x_0



- At time t=1 s:
  - The peak still occurs when the argument of f is x_0
  - But since the time is not 0
    → the function will be shifted in x by "vt"=1 cm
  - Position of the max: $x_1=x_0-1$



$f = f(x+vt)$ represents a wave traveling in the -x direction with velocity v

EM waves

- Wave equation: $\vec{\nabla}^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$
- Solution: $f = f(x \pm vt)$
  - Any function of argument $x \pm vt$
  - These solution represent waves traveling with velocity v
    - $x - vt$ represents a wave traveling in the +x direction
    - $x + vt$ represents a wave traveling in the –x direction
- Maxwell's equation: $\vec{\nabla}^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$
  - Same equation! Only difference: v=c
  - Solution: EM waves traveling with speed of light

→ The light IS an EM wave!!!

EM waves in SI

- This same result looks much more interesting in SI.
- Maxwell's equations in SI:

$$\vec{\nabla}^2 f = \mu_0 \varepsilon_0 \frac{\partial^2 f}{\partial t^2}$$

where ε_0 is the permittivity of free space
and μ_0 is the permeability of free space

- Maxwell's equations tell us what the velocity of an EM wave is:

$$v = 1/\sqrt{\mu_0 \varepsilon_0}$$

- ε_0 and μ_0 can be measured → we can predict velocity of EM waves:

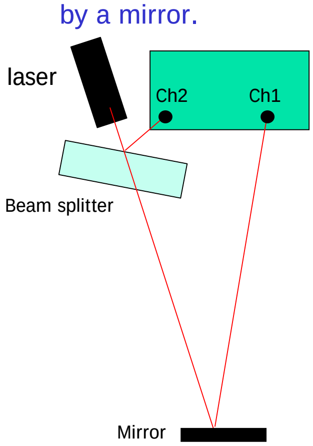
$$\epsilon_0 = 8.85418 \times 10^{-12}\,\text{Coulomb}^2\ \text{Newton}^{-1}\ \text{meter}^{-2}$$
$$\mu_0 = 4\pi \times 10^{-7}\,\text{Newton sec}^2\ \text{Coulomb}^{-2}$$

→ v=2.998 10^8 m/s^2 which is the speed of light!

- Maxwell was the first to realize that E&M equations were leading to a wave equation that was propagating at the speed of light: light is an EM wave!

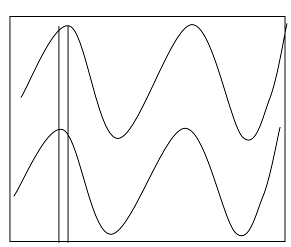
How to measure c (demo A4)

- Experimental setup: a neon laser beam is sent into a beam splitter. Part of it is reflected and part of it is refracted first and then reflected by a mirror.



- Difference in path between the 2 beams: ~ 17.15 m x 2 = 34.3 meters
- Measure the delay of channel 2 wrt channel 1 on the scope: 116 ns

→ v=34.3 m / 116 ns = 2.96 10^8 m/s



Summary and outlook

- Today:
  - Complete Maxwell's equations
    - The missing term leads to displacement currents
  - Solution of Maxwell's equations in vacuum
    - Wave equation → light is an EM radiation
- Next time:
  - Properties of EM radiation
  - Polarization and scattering of light