8.022 (E&M) – Lecture 15

Topics:

- More on Electromagnetic Inductance
  - Mutual and self inductance
  - Practical applications

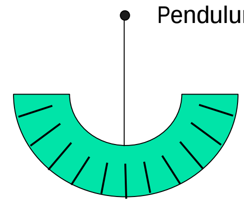
Last time

- Electromagnetic inductance
  - Faraday's (and Lentz's) law:
    - Integral form: $e.m.f. = -\frac{1}{c}\frac{\partial \Phi_B}{\partial t}$
    - Differential form: $\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$
- Let's elaborate a bit more on this important law…

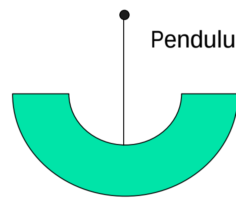
Cu pendulum in B field (H13)

- A copper pendulum is oscillating
  - Application of Lentz's law
- Turn on the magnetic field for the following 3 different situations:
  - Pendulum #1:
    - B crosses area with cuts
      - No effect
    - B crosses area above cuts
      - Stops slowly: Lentz's law
  - Pendulum #2:
    - No cuts in Cu
      - Stops abruptly: Lentz's law

Pendulum #1



Pendulum #2



Three ways of creating e.m.f.

- Faraday's law can be used to build generators:

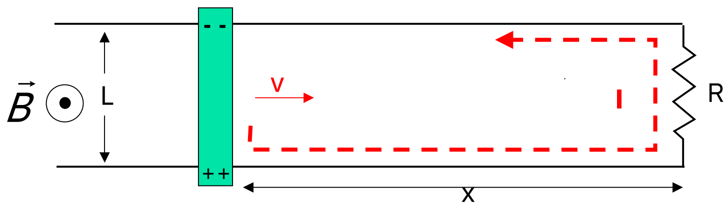
$$e.m.f. = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{a}$$

- 3 ways of creating e.m.f.:
  - Vary the area: S=S(t)
  - Vary the angle between **B** and d**a**
  - Vary magnitude of B: B=B(t)

$$e.m.f. = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{a}$$

Changing the area

- Sliding rod on rails:

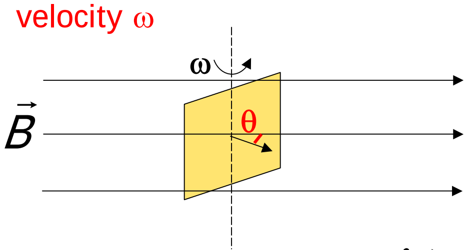


- As derived last week: $e.m.f. = \frac{vBL}{c}$
- Because of Lentz's law, direction of current is counterclockwise to oppose the change of flux of B
- Demo H4:
  - Loop + light bulb moving in B created by electromagnet

$$e.m.f. = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{a}$$

Changing angle between B and S

- Constant B and loop rotating around its axis with angular velocity ω



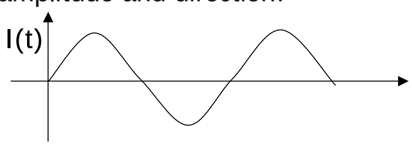
- If S is the area of the loop: $\int_S \vec{B} \bullet d\vec{a} = BS\cos\theta = BS\cos\omega t$

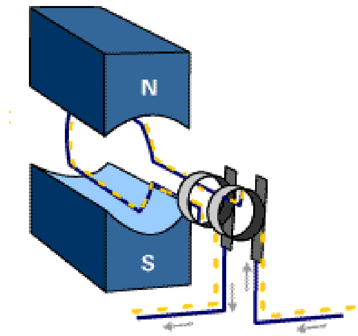
$$\Rightarrow |e.m.f.| = \frac{1}{c}\frac{\partial}{\partial t}(BS\cos\omega t) = \frac{\omega}{c}BS\sin\omega t$$

- This is an easy way to build an AC power generator

DC vs. AC current

- DC current
  - Electrons flow all in the same direction at the same rate
- AC current
  - The flow of electron varies with time in amplitude and direction:





- DC/AC generator
  - Uses DC to power electromagnet and induce AC on rotating loop
  - Why AC? Easier to step up and down for efficient transportation

$$e.m.f. = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B}\bullet d\vec{a}$$

Changing magnitude of B

- Suppose you have a way to vary over time the magnitude of B: B=B(t)
  - Flux of B: $\Phi_B = \int_S \vec{B}(t)\bullet d\vec{a} = B(t) S \cos\theta$
  - Generated e.m.f.: $|e.m.f.| = \frac{1}{c}\frac{\partial}{\partial t}\Phi_B = \frac{1}{c} S \frac{\partial B(t)}{\partial t}$
- How to created B=B(t)?
  - Loop of wire: $B \propto I$
  - If I=I(t) → B=B(t)

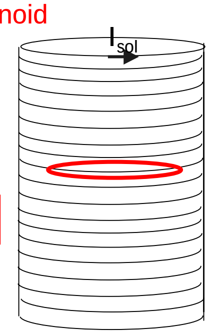
→ AC in a solenoid will do the trick!

Induced e.m.f.

- Consider a loop of wire with radius r inside a long solenoid
  - Solenoid:
    - N=# of loops, l=total length → n=N/l
    - $I_{sol} = I_{sol}(t)$
- What is the e.m.f. generated in the loop?
  - Find B inside solenoid: $B_{sol} = \frac{4\pi n I_{sol}(t)}{c}$

Q: can you derive this in 60 sec?

  - E.m.f. generated in loop:

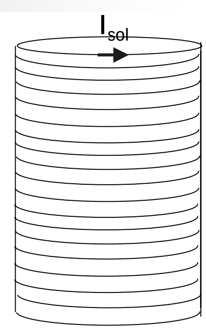


$$\left|e.m.f.\right| = \frac{1}{c}\frac{\partial}{\partial t}\Phi_B = \frac{1}{c}(\pi r^2)\frac{\partial B(t)}{\partial t} = \frac{4\pi^2 n r^2}{c^2}\frac{\partial I_{sol}(t)}{\partial t}$$

→ The e.m.f. will depend by the geometry of the setup and on the rate of change of the I over time

Induced e.m.f. on solenoid itself

- What if the "loop" is the solenoid itself?
  - Will any e.m.f. be created?
- Remember Faraday's law: $e.m.f. = -\frac{1}{c}\frac{\partial}{\partial t}\int_S \vec{B} \bullet d\vec{a}$
  - B inside solenoid: $B_{sol} = \frac{4\pi n I_{sol}(t)}{c}$
  - Flux of B through each loop: $\Phi_B^{1\,loop} = BS_{1\,loop} = \frac{4\pi n I_{sol}(t)}{c}\pi R^2$
  - Flux of B through N loops: $\Phi_B^{Tot} = N\Phi_B^{1\,loop} = \frac{4\pi^2 R^2 N^2}{c\, l} I_{sol}(t)$



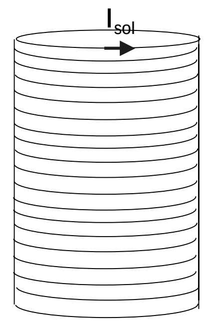
→ Induced e.m.f. on solenoid: $\left|e.m.f.\right| = \frac{4\pi^2 R^2 N^2}{c^2 l}\frac{\partial I_{sol}(t)}{\partial t}$

Back e.m.f.

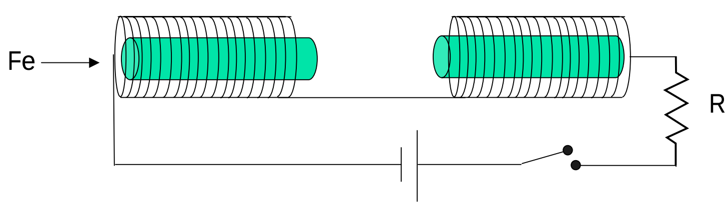
- Magnitude of induced e.m.f. on solenoid:

$$\left|e.m.f.\right| = \frac{4\pi^2 R^2 N^2}{c^2 l}\frac{\partial I_{sol}(t)}{\partial t}$$

- How about the direction? And the effect?
  - Use Lentz's law to predict direction of induced current
    - If I_{sol} increases → B increases → flux increases
      I_{loop} will fight change → opposite direction as I_{sol}
    - If I_{sol} decreases → B decreases → flux decreases
      I_{loop} will fight change → same direction as I_{sol}
- Conclusion:
  The inductance always opposes the change in the current
  The e.m.f. created is called back e.m.f. as it acts back on the circuit trying to oppose changes



Example of back e.m.f. (H17)



- Close switch: wire jumps → I flows (30 A)
- Open switch: big spark due by back emf

Self Inductance L

- Self-induced e.m.f. in the solenoid:

$$\left|e.m.f.\right| = \frac{4\pi^2 R^2 N^2}{c^2 l}\frac{\partial I_{sol}(t)}{\partial t} \Rightarrow \boxed{\left|e.m.f.\right| = L\frac{\partial I_{sol}(t)}{\partial t}}$$

- Let's examine this in detail:
  - e.m.f. depends on change over time of current: dI/dt
  - A bunch of constants depending on geometry called self inductance L
- For a solenoid: $L_{sol} = \dfrac{4\pi^2 R^2 N^2}{c^2 l}$
- Units:
  - cgs: $[L] = \dfrac{[e.m.f.]}{[current]/[time]} = \dfrac{esu/cm}{(esu/s)/s} = \dfrac{\sec^2}{cm}$
  - SI: $[L] = \dfrac{[e.m.f.]}{[current]/[time]} = \dfrac{V}{A/s} \equiv Henry\,(H)$

Energy stored in inductors

- Consider an inductor L in which we start flowing a current I
  - As soon as the current starts flowing, a back-emf tries to fight this current back
  - Power needed to fight the back-emf:

$$P = I \times e.m.f. = IL\frac{\partial I}{\partial t}$$

- Calculate work to increase the current from 0 → I when t: 0 → t

$$W = \int_{t=0}^{t} P dt = \int_{t=0}^{t} LI\frac{\partial I}{\partial t}dt = L\int_{I=0}^{I} IdI = \frac{1}{2}LI^2$$

- Energy stored in the inductor: $\boxed{W = \frac{1}{2}LI^2}$

How is energy stored in inductors?

- We created a magnetic field where there was none: work necessary to create the magnetic field is the energy stored in the B itself
  - Same as energy stored in electric field of a capacitor
  - Not surprising: special relativity!
- Energy density of magnetic field (solenoid example)
  - Energy stored in solenoid: $U_L=LI^2/2$
  - Self inductance of a solenoid: $L=4\pi^2R^2N^2/lc^2$
  - B created by solenoid: $B=4\pi N/lc$

→ $$U_L = \frac{1}{2}LI^2 = \frac{1}{2}\frac{4\pi^2N^2}{c^2l}I^2 = \frac{1}{8\pi}\left(\pi R^2 l\right)\left(\frac{4\pi N}{cl}I\right)^2 = Volume\,\frac{B^2}{8\pi}$$

- Energy density of B: $u_B = \frac{B^2}{8\pi}$
  - Similar to energy density of the electric field: $u_E = \frac{E^2}{8\pi}$

How do we calculate L in psets?

Just some examples…

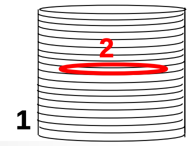
- Strategy 1:
  - L is the proportionality constant between induced emf and variation over time of current:

$$\left|e.m.f.\right| = L\frac{\partial I(t)}{\partial t}$$

- Strategy 2:
  - Exploit the fact that energy stored in the magnetic field is the energy stored in the inductor:

$$\text{Energy stored in B} = \int_V \frac{B^2}{8\pi}dV = \frac{1}{2}LI^2$$

Mutual inductance



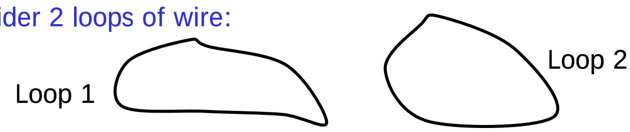
- Back to the loop inside the solenoid
  - Label solenoid with 1 and loop with 2
- e.m.f. induced on loop (ε_2) depends on dI_1/dt and constant M_{21}

$$\varepsilon_2 = M_{21} \frac{\partial I_1}{\partial t}$$

  where M_{21} is the coefficient of mutual inductance
  - For this particular configuration we already calculated that $M_{21} = \frac{4\pi^2 r^2 N}{c^2 l}$
- Now do the opposite: run a current $I_2(t)$ in the loop and calculate e.m.f. induced on solenoid (ε_1): $\varepsilon_1 = M_{12} \frac{\partial I_2}{\partial t}$
- How to calculate M_{12}???
  - No need to calculate it! Reciprocity theorem: $M_{12}=M_{21}$

Reciprocity theorem

- Consider 2 loops of wire:



- Current I runs through loop 1. What is Φ_B through loop 2 due to 1?

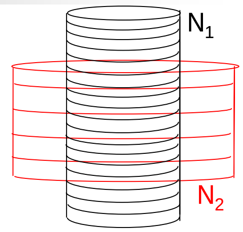
$$\Phi_{21}^B = \int_{S_2} \vec{B}_1 \bullet d\vec{a}_2$$

- Now rewrite this result in terms of vector potential and use Stokes:

$$\Phi_{21} = \int_{S_2} \vec{B}_1 \bullet d\vec{a}_2 = \int_{S_2} \left(\nabla \times \vec{A}_1\right) \bullet d\vec{a}_2 = \int_{C_2} \vec{A}_1 \bullet d\vec{l}_2$$

- Since $\vec{A}_1 = \frac{I}{c} \oint_{C_1} \frac{d\vec{l}_1}{r}$ we obtain $\Phi_{21} = \frac{I}{c} \int_{C_2} \int_{C_1} \frac{d\vec{l}_1}{r} \bullet d\vec{l}_2 = \Phi_{12}$
- Same fluxes→ if currents are the same: $M_{12}=M_{21}$

Transformers



- Devices to step up (or down) AC currents
  - Practical application of mutual inductance
- Simplest implementation:
  - Primary solenoid (black): N_1 turns
  - Secondary solenoid (red): N_2 turns
- I(t) in the primary will induce a varying Φ_B through itself:

$$\varepsilon_1 = \frac{N_1}{c}\frac{d\Phi_B}{dt}$$

  - where Φ_B=magnetic flux through single turn
- Flux is the same in second solenoid → induced e.m.f. is: $\varepsilon_2 = \frac{N_2}{c}\frac{d\Phi_B}{dt}$
- Comparing: $\varepsilon_2 = \varepsilon_1 \frac{N_2}{N_1}$

Depending on number of turns we can
- increase voltage ($N_2>N_1$)
- reduce the voltage ($N_2<N_1$)

Demos on mutual inductance

- Single turn around primary coil (H10)
  - Emf: 208 V AC
  - Primary coil: N_1= 220 turns
  - Secondary coil: N_2= 1 turn
  - Effect: V goes down, but current goes up and melts the nail!
  - Explanation: Power = VI is conserved between the 2 coils
- Variable turns around primary coil (H9)
  - Same primary; show how current in secondary goes as we add loops
- High turn secondary (H11)
  - Emf: 208 V AC
  - Primary coil: N_1= 220 turns
  - Secondary coil: N_2= 10,000 turn
  - Effect: Small currents, but very large V will cause big sparks!

Summary and outlook

- Today:
  - Self inductance
    - Energy stored in inductor
  - Mutual inductance
    - And its applications: transformers
- Next time:
  - Inductors in circuits
- Quiz II-preparation supplies available here!