Massachusetts Institute of Technology
Physics 8.022 – Fall 2004
Quiz #1

- **No books, notes or calculators allowed.** All needed formulae are given below.
- Please **box your final answer** and **specify magnitude AND direction for all vectors**.
- Total points in the quiz: 100. Not all problems receive equal credit. **Work on the problems you are more comfortable with first!**

Useful Formulae

Conservative Field: $\int_C F \cdot d\vec{r} = 0$ for any closed path C, $W_{ab,C} = W_{ab,C'}$ for any C, C' connecting a and b, $\vec{F} = -\vec{\nabla} U$, $\vec{\nabla} \times \vec{F} = 0$

Coulomb Law: $\vec{F}_{21} = \frac{q_1 q_2}{r^2}\hat{r}_{21}$ for two point charges at distance r. $\vec{F}_{12} = -\vec{F}_{21}$, and for charges dq_1 and dq_2 that make part of continuous charge distributions 1 and 2, $d\vec{F}_{21} = \frac{dq_1 dq_2}{r^2}\hat{r}_{21}$

Electric Field: *at point 2* due to q_1 $\vec{E}_1 = \frac{q_1}{r^2}\hat{r}_{21}$. If q_1 is not a point charge but part of a continuous distribution, $d\vec{E} = \frac{dq}{r^2}\hat{r}$

Principle of Superposition: Two or more electric fields acting at a given point P add vectorially: $\vec{E}_P = \vec{E}_1 + \vec{E}_2 + ... + \vec{E}_n$

Electrostatic Field is Conservative: $\vec{\nabla} \times \vec{E} = 0$ and thus there exists scalar function ϕ such that $\vec{E} = -\vec{\nabla}\phi$ where $d\phi = \frac{dq}{r}$

Electrostatic potential: The potential at $\vec{x}$ with respect to a *ref* point is $\phi(\vec{x}) - \phi(ref) = -\int_{ref}^{\vec{x}} \vec{E} \cdot d\vec{r} = -\frac{W_{ref \to \vec{x}}}{q}$

Gauss Law: $\int_S \vec{E} \cdot d\vec{a} = 4\pi \int_V \rho dV$ where S is a closed surface and V is its corresponding volume (integral form) or $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ (differential form).

Poisson Eqn: $\nabla^2 \phi = -4\pi\rho$, Laplace Eqn: $\nabla^2\phi = 0$

Energy: $U = \frac{1}{2}\int_V dV \int_{V'} dV' \frac{\rho(\vec{x})\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{2}\int_V \rho\phi dV = \frac{1}{8\pi}\int_V E^2 dV$

Electric Force on Conductors: $\frac{dF}{da} = 2\pi\sigma^2 = \frac{E^2}{8\pi}$

Capacitance: $Q = CV$, (energy) $U = \frac{1}{2}CV^2$

Capacitor networks: Parallel: $C = C_1 + C_2$, series: $\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$

Gradient: in cartesian $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}$, in cylindrical $\vec{\nabla} f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{z}$, in spherical $\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}$

Divergence: in cartesian $\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$, in cylindrical $\vec{\nabla} \cdot \vec{F} = \frac{F_\rho}{\rho} + \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$, in spherical $\vec{\nabla} \cdot \vec{F} = \frac{2F_r}{r} + \frac{\partial F_r}{\partial r} + \frac{F_\theta}{r}cot\theta + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r sin\theta}\frac{\partial F_\phi}{\partial \phi}$

Binomial expansion:

$$(1 \pm x)^n = 1 \pm \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} \pm \cdots (x^2 < 1); \quad (1 \pm x)^{-n} = 1 \mp \frac{nx}{1!} + \frac{n(n+1)x^2}{2!} \mp \cdots (x^2 < 1)$$

Problem 1: Electric charges and fields (25 points)

An electrostatic potential in the 3-dimensional Euclidean space is given by

$$\phi = \frac{Q_0}{a}(1 - \frac{z}{3a} - \frac{a}{r}) \ \ for \ \ 0 < r < a, \ \ \phi = -\frac{Q_0 a z}{3r^3} \ \ for \ \ r > a$$

where $\vec{r} = r\hat{r} = x\hat{x} + y\hat{y} + z\hat{z}$ while Q_0 and a are positive definite constants of appropriate dimensions. The accompanying electrostatic field derived from the gradient of ϕ $(i.e., \vec{E} = -\vec{\nabla}\phi)$ is found to be

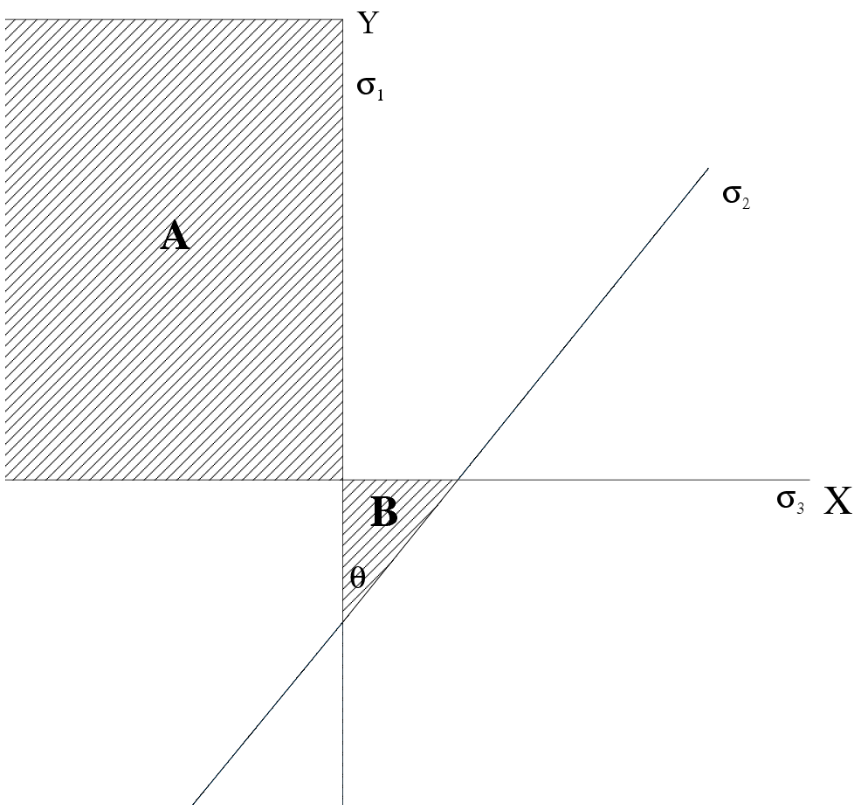
$$\vec{E} = -\hat{r}\frac{Q_0}{a^2}(\frac{a^2}{r^2} - \frac{cos\theta}{3}) - \hat{\theta}\frac{Q_0 sin\theta}{3a^2} \ \ for \ \ 0 < r < a,$$
$$\vec{E} = -\hat{r}\frac{2Q_0 acos\theta}{3r^3} - \hat{\theta}\frac{Q_0 asin\theta}{3r^3} \ \ for \ \ r > a,$$

where $z = rcos\theta$. Our goal is to find the electric charges that set up this field by following the next steps. **Please box your final answers in each part.**

1. Is ϕ continuous at $r = a$? Was that expected?
2. Is $\vec{E}$ continuous at $r = a$? What does this tell us for a spherical shell at a?
3. Find ρ, the volume electric charge density ($\rho = dq/dv$) anywhere for $r < a$.
4. Now find ρ for $r > a$.
5. Consider a sphere of radius $R < a$ and find the total charge Q_1 inside it.
6. Is your answer in the previous part consistent with your answer in part (3)? Where can Q_1 be located?
7. Now consider a sphere of radius $R > a$ and find the total charge Q_2 inside it. Find Q_3=Q_2-Q_1.
8. Given your answer in parts (3) and (4), where can Q_3 be located?
9. Find the surface charge density everywhere on a sphere of radius a. In order to do this, perform a Gaussian experiment using an infinitesimal cylinder positioned with its axis radially and centered onto the sphere of radius a.
10. Associate the various terms of the charge distributions with the given electric potential ϕ inside and outside of $r = a$.

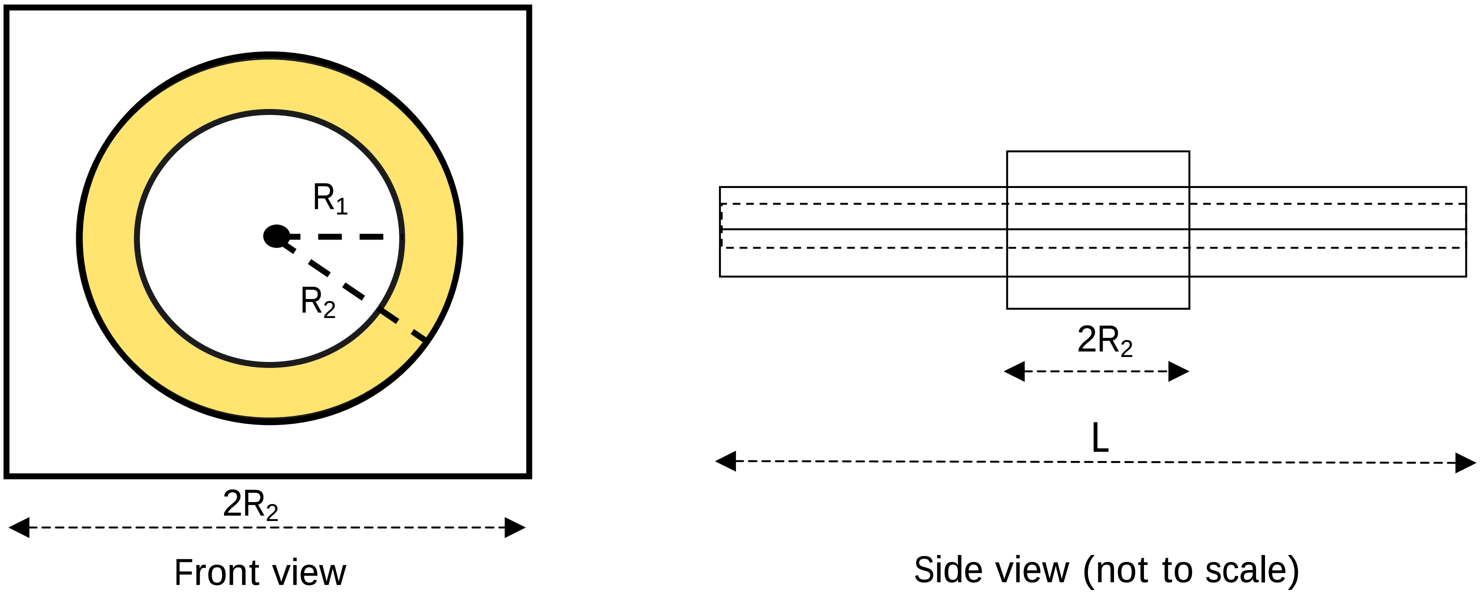
Problem 2: Infinite charged sheets (25 points)

Three infinite charged **non conductive** sheets are arranged as in the figure below. Each sheet carries surface charge density of $\sigma_1 = \sigma_0$, σ_2 and σ_3 . Only σ_0 and the angle θ between the two planes carrying the σ_1 and σ_2 charged sheets are known. **Please box your final answers in each part.**



a) Find σ_2 and σ_3 such that the electric field inside the enclosed triangular region marked (B) in the figure is zero (express all your answers in terms of σ_0 and θ).
b) Find the energy density (energy per unit volume) of the electric field in the region marked (A) in the figure (express all your answers in terms of σ_0 and θ).
c) If the sheet carrying surface charge density σ_2 is moved downward along the y axis does the total energy of the system increase of decrease?
d) If the sheets were released from the positions shown, how would they move? Hint: think about your answer to part (3)

Problem # 3: cylindrical capacitor (35 points)



A charge +Q is deposited on a **conductive wire** of length L and radius R_0.
An electrically neutral, **conductive cylindrical shell** of inner radius R_1, outer radius R_2 and length L is symmetrically positioned around the wire (see figure).
NB: in this problem $R_0 << R_1 \sim R_2$ are $<<$ L.

Calculate:

a) The line charge density on the wire (λ)
b) The surface charge density on the inner (σ_{inner}) and outer (σ_{outer}) shell of the cylinder and the volume charge density (ρ) inside the cylindrical shell, for $R_1<r<R_2$. NB: always specify the sign of the charge.
c) The electric field $\vec{E}$ everywhere in space ($r<R_0$, $R_0<r<R_1$, $R_1<r<R_2$, $r>R_2$)
d) The flux of the electric field (Φ_E) through a cube of side $2R_2$ centered on the center of the wire (see figure).

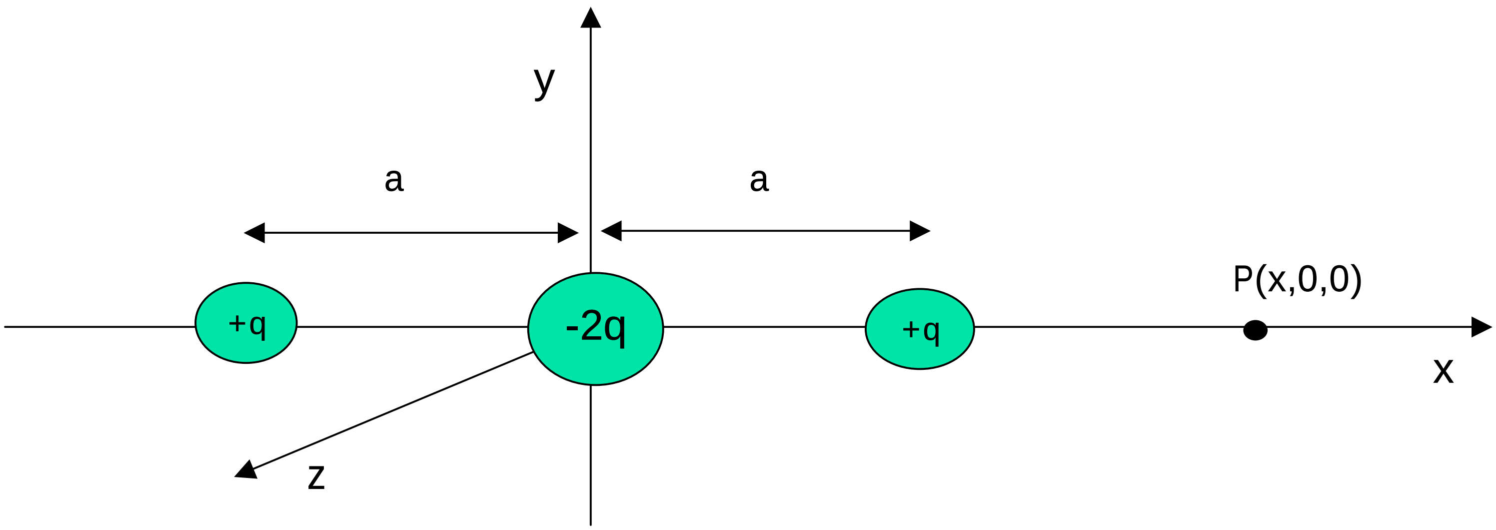
Now deposit a charge –Q on the cylindrical shell.
Calculate:

e) The surface charge density on the inner (σ'_{inner}) and outer (σ'_{outer}) shell of the cylinder and the volume charge density (ρ') inside the cylindrical shell, for $R_1<r<R_2$. NB: always specify the sign of the charge.
f) The electric field $\vec{E}\,'$ everywhere in space
g) The difference in potential V between the cylinder and the wire: $V= \phi_{cylinder} - \phi_{wire}$
h) What is the capacitance C of the system (wire + cylindrical shell)?
i) What is the energy stored in the system?
j) What is the capacitance C' when we double the charge on the wire (+2Q) and on the cylinder (-2Q)?

Please box your final answers in each part.

Problem #4: electric quadrupole (15 points)

Two positive charges (+q each) and one negative charge (-2q) are positioned as shown in the following figure: the negative charge is in the center of the coordinate system while the two positive charges are at coordinates (–a,0,0) and (+a,0,0) respectively.



Calculate:

a) Electric potential ϕ created by the distribution of charges in the point P(x,0,0)

b) Electric field $\vec{E}$ in P(x,0,0)

c) What is the asymptotic behavior of the electric field when |x|>>a? (e.g. E~ const, 1/x, $1/x^2$, $1/x^3$, ln x, …)

Please box your final answers in each part.