Example 8.12 Free Fall with Air Drag

Consider an object of mass m that is in free fall but experiencing air resistance. The magnitude of the drag force is given by Eq. (8.6.1), where ρ is the density of air, A is the cross-sectional area of the object in a plane perpendicular to the motion, and C_D is the drag coefficient. Assume that the object is released from rest and very quickly attains speeds in which Eq. (8.6.1) applies. Determine (i) the terminal velocity, and (ii) the velocity of the object as a function of time.

Solution: Choose positive y -direction downwards with the origin at the initial position of the object as shown in Figure 8.48(a).

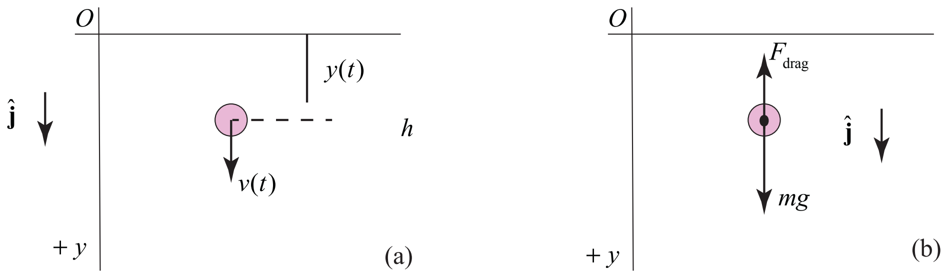

Figure 8.48 (a) Coordinate system for marble; (b) free body force diagram on marble

There are two forces acting on the object: the gravitational force, and the drag force which is given by Eq. (8.6.1). The free body diagram is shown in the Figure 8.48(b). Newton's Second Law is then

$$mg-(1/2)C_D A\rho v^2 = m\frac{dv}{dt}, \tag{8.6.109}$$

Set $\beta=(1/2)C_D A\rho$. Newton's Second Law can then be written as

$$mg-\beta v^2 = m\frac{dv}{dt}. \tag{8.6.110}$$

Initially when the object is just released with $v=0$, the air drag is zero and the acceleration dv/dt is maximum. As the object increases its velocity, the air drag becomes larger and dv/dt decreases until the object reaches terminal velocity and $dv/dt=0$. Set $dv/dt=0$ in Eq. (8.6.15) and solve for the terminal velocity yielding.

$$v_\infty = \sqrt{\frac{mg}{\beta}} = \sqrt{\frac{2mg}{C_D A\rho}}. \tag{8.6.111}$$

Values for the magnitude of the terminal velocity is shown in Table 8.3 for a variety of objects with the same drag coefficient $C_D=0.5$.

Table 8.3 Terminal Velocities for Different Sized Objects with $C_D = 0.5$

Object	Mass m (kg)	Area A (m^2)	Terminal Velocity v_∞ $(\text{m}\cdot\text{s}^{-1})$
Rain drop	4×10^{-6}	3×10^{-6}	6.5
Hailstone	4×10^{-3}	3×10^{-4}	20
Osprey	20	2.5×10^{-1}	50
Human Being	7.5×10^{1}	6×10^{-1}	60

In order to integrate Eq. (8.6.15), we shall apply the technique of separation of variables and integration by partial fractions. First rewrite Eq. (8.6.15) as

$$\frac{-\beta}{m}dt = \frac{dv}{\left(v^2 - \dfrac{mg}{\beta}\right)} = \frac{dv}{\left(v^2 - v_\infty^2\right)} = \left(-\frac{1}{2v_\infty(v+v_\infty)} + \frac{1}{2v_\infty(v-v_\infty)}\right)dv\,. \quad (8.6.112)$$

An integral expression of Eq. (8.6.112) is then

$$-\int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'+v_\infty)} + \int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'-v_\infty)} = -\frac{\beta}{m}\int_{t'=0}^{t'=t} dt'\,. \quad (8.6.113)$$

Integration yields

$$\begin{aligned}
-\int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'+v_\infty)} + \int_{v'=0}^{v'=v(t)} \frac{dv'}{2v_\infty(v'-v_\infty)} &= -\frac{\beta}{m}\int_{t'=0}^{t'=t} dt' \\
\frac{1}{2v_\infty}\left(-\ln\left(\frac{v(t)+v_\infty}{v_\infty}\right) + \ln\left(\frac{v_\infty - v(t)}{v_\infty}\right)\right) &= -\frac{\beta}{m}t
\end{aligned}\,. \quad (8.6.114)$$

After some algebraic manipulations, Eq. (8.6.114) can be rewritten as

$$\ln\left(\frac{v_\infty - v(t)}{v(t)+v_\infty}\right) = -\frac{2v_\infty\beta}{m}t \quad (8.6.115)$$

Exponentiate Eq. (8.6.115) yields

$$\left(\frac{v_\infty - v(t)}{v(t)+v_\infty}\right) = e^{-\frac{2v_\infty\beta}{m}t}\,. \quad (8.6.116)$$

After some algebraic rearrangement the y-component of the velocity as a function of time is given by

$$v(t) = v_\infty \left(\frac{1 - e^{-\frac{2v_\infty \beta}{m}t}}{1 + e^{-\frac{2v_\infty \beta}{m}t}} \right) = v_\infty \tan h\left(\frac{v_\infty \beta}{m} t \right). \qquad (8.6.117)$$

where $\dfrac{v_\infty \beta}{m} = \dfrac{\beta}{m}\sqrt{\dfrac{mg}{\beta}} = \sqrt{\dfrac{\beta g}{m}} = \sqrt{\dfrac{(1/2)C_D A \rho g}{m}}$.