### 8.5.2 Continuous Systems and Newton's Second Law as a Differential Equations

We can determine the tension at a distance $y$ from the ceiling in Example 8.4, by an alternative method, a technique that will generalize to many types of "continuous systems". Choose a coordinate system with the origin at the ceiling and the positive $y$-direction pointing downward as in Figure 8.25. Consider as the system a small element of the rope between the points $y$ and $y+\Delta y$. This small element has length $\Delta y$, The small element has mass $\Delta m = (M/L)\Delta y$ and is shown in Figure 8.29.

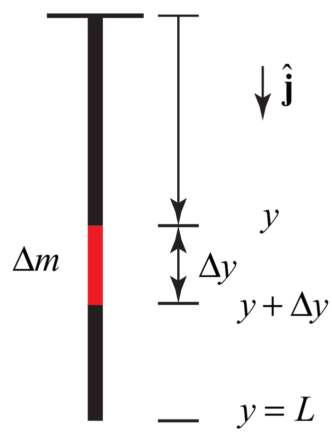


**Figure 8.29** Small mass element of the rope

The forces acting on the small element are the tension, $T(y)$ at $y$ directed upward, the tension $T(y+\Delta y)$ at $y+\Delta y$ directed downward, and the gravitational force $\Delta m g$ directed downward. The tension $T(y+\Delta y)$ is equal to the tension $T(y)$ plus a small difference $\Delta T$,

$$T(y+\Delta y) = T(y) + \Delta T \,. \qquad (8.5.15)$$

The small difference in general can be positive, zero, or negative. The free body force diagram is shown in Figure 8.30.

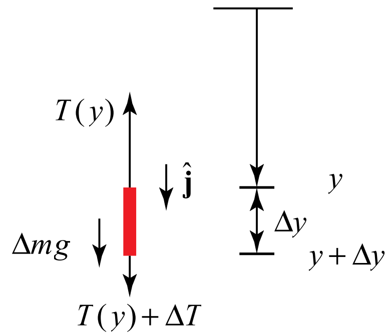


**Figure 8.30** Free body force diagram on small mass element

Now apply Newton's Second Law to the small element

$$\Delta mg + T(y) - (T(y) + \Delta T) = 0 \qquad (8.5.16)$$

The difference in the tension is then $\Delta T = -\Delta mg$. We now substitute our result for the mass of the element $\Delta m = (M / L)\Delta y$, and find that that

$$\Delta T = -(M / L)\Delta y g \ . \qquad (8.5.17)$$

Divide through by $\Delta y$, yielding $\Delta T / \Delta y = -(M / L)g$. Now take the limit in which the length of the small element goes to zero, $\Delta y \to 0$,

$$\lim_{\Delta y \to 0} \frac{\Delta T}{\Delta y} = -(M / L)g \ . \qquad (8.5.18)$$

Recall that the left hand side of Eq. (8.5.18) is the definition of the derivative of the tension with respect to $y$, and so we arrive at Eq. (8.5.14),

$$\frac{dT}{dy} = -(M / L)g \, .$$

We can solve the differential equation, Eq. (8.5.14), by a technique called **separation of variables**. We rewrite the equation as $dT = -(M / L)gdy$ and integrate both sides. Our integral will be a definite integral in which we integrate a 'dummy' integration variable $y'$ from $y' = 0$ to $y' = y$ and the corresponding $T'$ from $T' = T(y = 0)$ to $T' = T(y)$:

$$\int_{T'=T(y=0)}^{T'=T(y)} dT' = -(M / L)g \int_{y'=0}^{y'=y} dy' \ . \qquad (8.5.19)$$

After integration and substitution of the limits, we have that

$$T(y) - T(y=0) = -(M / L)gy \ . \tag{8.5.20}$$

Us the fact that tension at the top of the rope is $T(y=0) = Mg$ and find that

$$T(y) = Mg(1 - y / L)$$

in agreement with our earlier result, Eq. (8.5.13).