

## 19.6 Angular Momentum of a System of Particles

We now calculate the angular momentum about the point  $S$  associated with a system of  $N$  point particles. Label each individual particle by the index  $j$ ,  $j=1,2,\dots,N$ . Let the  $j^{\text{th}}$  particle have mass  $m_j$  and velocity  $\vec{v}_j$ . The momentum of an individual particle is then  $\vec{p}_j = m_j \vec{v}_j$ . Let  $\vec{r}_{S,j}$  be the vector from the point  $S$  to the  $j^{\text{th}}$  particle, and let  $\theta_j$  be the angle between the vectors  $\vec{r}_{S,j}$  and  $\vec{p}_j$  (Figure 19.10).

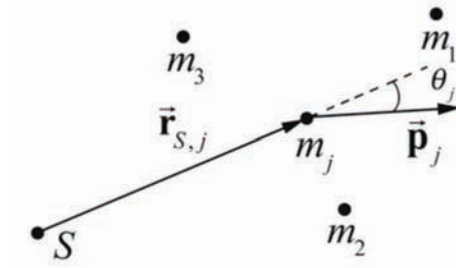


Figure 19.10 System of particles

The angular momentum  $\vec{L}_{S,j}$  of the  $j^{\text{th}}$  particle is

$$\vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j. \quad (19.5.1)$$

The angular momentum for the system of particles is the vector sum of the individual angular momenta,

$$\vec{L}_S^{\text{sys}} = \sum_{j=1}^{j=N} \vec{L}_{S,j} = \sum_{j=1}^{j=N} \vec{r}_{S,j} \times \vec{p}_j. \quad (19.5.2)$$

The change in the angular momentum of the system of particles about a point  $S$  is given by

$$\frac{d\vec{L}_S^{\text{sys}}}{dt} = \frac{d}{dt} \sum_{j=1}^{j=N} \vec{L}_{S,j} = \sum_{j=1}^{j=N} \left( \frac{d\vec{r}_{S,j}}{dt} \times \vec{p}_j + \vec{r}_{S,j} \times \frac{d\vec{p}_j}{dt} \right). \quad (19.5.3)$$

Because the velocity of the  $j^{\text{th}}$  particle is  $\vec{v}_{S,j} = d\vec{r}_{S,j}/dt$ , the first term in the parentheses vanishes (the cross product of a vector with itself is zero because they are parallel to each other)

$$\frac{d\vec{r}_{S,j}}{dt} \times \vec{p}_j = \vec{v}_{S,j} \times m_j \vec{v}_{S,j} = 0. \quad (19.5.4)$$

Substitute Eq. (19.5.4) and  $\vec{F}_j = d\vec{p}_j / dt$  into Eq. (19.5.3) yielding

$$\frac{d\vec{L}_S^{\text{sys}}}{dt} = \sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \frac{d\vec{p}_j}{dt} \right) = \sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \vec{F}_j \right). \quad (19.5.5)$$

Because

$$\sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \vec{F}_j \right) = \sum_{j=1}^{j=N} \vec{\tau}_{S,j} = \vec{\tau}_S^{\text{ext}} + \vec{\tau}_S^{\text{int}} \quad (19.5.6)$$

We have already shown in Chapter 17.4 that when we assume all internal forces are directed along the line connecting the two interacting objects then the internal torque about the point  $S$  is zero,

$$\vec{\tau}_S^{\text{int}} = \vec{0}. \quad (19.5.7)$$

Eq. (19.5.6) simplifies to

$$\sum_{j=1}^{j=N} \left( \vec{r}_{S,j} \times \vec{F}_j \right) = \sum_{j=1}^{j=N} \vec{\tau}_{S,j} = \vec{\tau}_S^{\text{ext}}. \quad (19.5.8)$$

Therefore Eq. (19.5.5) becomes

$$\vec{\tau}_S^{\text{ext}} = \frac{d\vec{L}_S^{\text{sys}}}{dt}. \quad (19.5.9)$$

The external torque about the point  $S$  is equal to the time derivative of the angular momentum of the system about that point.

### Example 19.5 Angular Momentum of Two Particles undergoing Circular Motion

Two identical particles of mass  $m$  move in a circle of radius  $R$ , with angular velocity  $\vec{\omega} = \omega_z \hat{k}$ ,  $\omega_z > 0$ ,  $\omega$  about the  $z$ -axis in a plane parallel to but a distance  $h$  above the  $x$ - $y$  plane. The particles are located on opposite sides of the circle (Figure 19.11). Find the magnitude and the direction of the angular momentum about the point  $S$  (the origin).

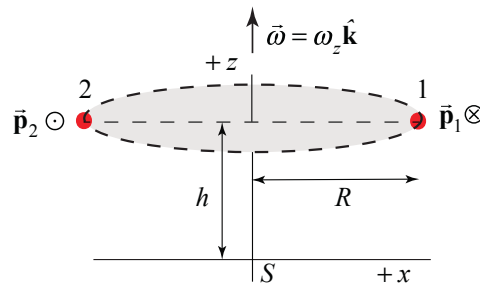
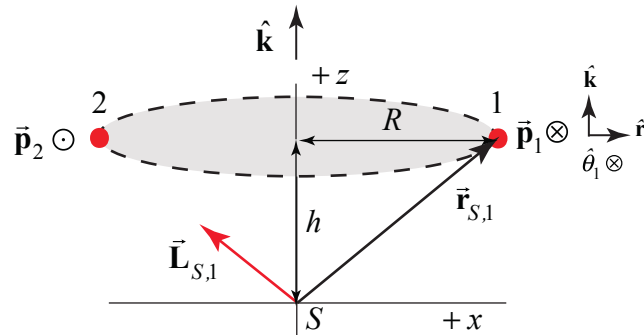
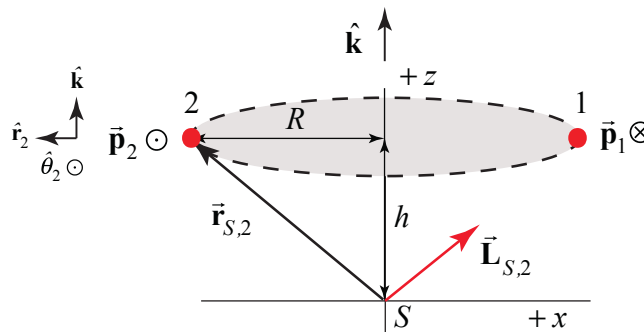


Figure 19.11 Example 19.5

**Solution:** The angular momentum about the origin is the sum of the contributions from each object. The calculation of each contribution will be identical to the calculation in Example 19.3



**Figure 19.12** Angular momentum of particle 1 about origin



**Figure 19.13** Angular momentum of particle 2 about origin

For particle 1 (Figure 19.12), the angular momentum about the point  $S$  is

$$\vec{L}_{S,1} = \vec{r}_{S,1} \times \vec{p}_1 = (R\hat{r}_1 + h\hat{k}) \times mR\omega_z\hat{\theta}_1 = mR^2\omega_z\hat{k} - hmR\omega_z\hat{r}_1.$$

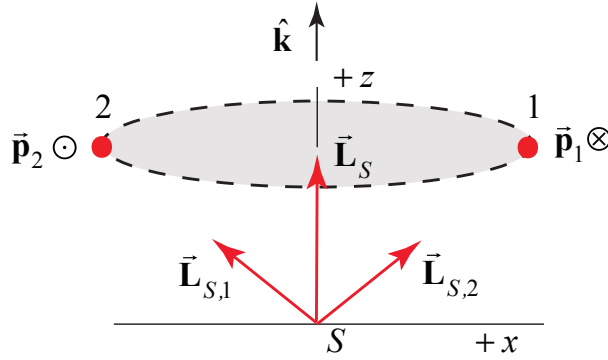
For particle 2, (Figure 19.13), the angular momentum about the point  $S$  is

$$\vec{L}_{S,2} = \vec{r}_{S,2} \times \vec{p}_2 = (R\hat{r}_2 + h\hat{k}) \times mR\omega_z\hat{\theta}_2 = mR^2\omega_z\hat{k} - hmR\omega_z\hat{r}_2.$$

Because the particles are located on opposite sides of the circle,  $\hat{r}_1 = -\hat{r}_2$ . The vector sum only points along the  $z$ -axis and is equal to

$$\vec{L}_S = \vec{L}_{S,1} + \vec{L}_{S,2} = 2mR^2\omega_z\hat{k}. \quad (19.5.10)$$

The two angular momentum vectors are shown in Figure 19.14.



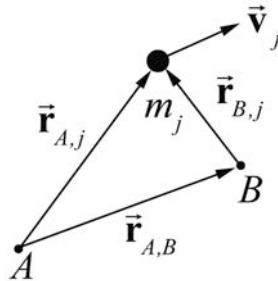
**Figure 19.14** Angular momentum about the point  $S$  of both particles and their sum

The moment of inertia of the two particles about the  $z$ -axis is given by  $I_S = 2mR^2$ . Therefore  $\vec{L}_S = I_S \vec{\omega}$ . The important point about this example is that the two objects are symmetrically distributed with respect to the  $z$ -axis (opposite sides of the circular orbit). Therefore the angular momentum about any point  $S$  along the  $z$ -axis has the same value  $\vec{L}_S = 2mr^2\omega\hat{k}$ , which is constant in magnitude and points in the  $+z$ -direction. This result generalizes to any rigid body in which the mass is distributed symmetrically about the axis of rotation.

### Example 19.6 Angular Momentum of a System of Particles about Different Points

Consider a system of  $N$  particles, and two points  $A$  and  $B$  (Figure 19.15). The angular momentum of the  $j^{\text{th}}$  particle about the point  $A$  is given by

$$\vec{L}_{A,j} = \vec{r}_{A,j} \times m_j \vec{v}_j. \quad (19.5.11)$$



**Figure 19.15** Vector triangle relating position of object and points  $A$  and  $B$

The angular momentum of the system of particles about the point  $A$  is given by the sum

$$\vec{L}_A = \sum_{j=1}^N \vec{L}_{A,j} = \sum_{j=1}^N \vec{r}_{A,j} \times m_j \vec{v}_j \quad (19.5.12)$$

The angular momentum about the point  $B$  can be calculated in a similar way and is given by

$$\vec{\mathbf{L}}_B = \sum_{j=1}^N \vec{\mathbf{L}}_{B,j} = \sum_{j=1}^N \vec{\mathbf{r}}_{B,j} \times m_j \vec{\mathbf{v}}_j. \quad (19.5.13)$$

From Figure 19.15, the vectors

$$\vec{\mathbf{r}}_{A,j} = \vec{\mathbf{r}}_{B,j} + \vec{\mathbf{r}}_{A,B}. \quad (19.5.14)$$

We can substitute Eq. (19.5.14) into Eq. (19.5.12) yielding

$$\vec{\mathbf{L}}_A = \sum_{j=1}^N (\vec{\mathbf{r}}_{B,j} + \vec{\mathbf{r}}_{A,B}) \times m_j \vec{\mathbf{v}}_j = \sum_{j=1}^N \vec{\mathbf{r}}_{B,j} \times m_j \vec{\mathbf{v}}_j + \sum_{j=1}^N \vec{\mathbf{r}}_{A,B} \times m_j \vec{\mathbf{v}}_j. \quad (19.5.15)$$

The first term in Eq. (19.5.15) is the angular momentum about the point  $B$ . The vector  $\vec{\mathbf{r}}_{A,B}$  is a constant and so can be pulled out of the sum in the second term, and Eq. (19.5.15) becomes

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \sum_{j=1}^N m_j \vec{\mathbf{v}}_j \quad (19.5.16)$$

The sum in the second term is the momentum of the system

$$\vec{\mathbf{p}}_{\text{sys}} = \sum_{j=1}^N m_j \vec{\mathbf{v}}_j. \quad (19.5.17)$$

Therefore the angular momentum about the points  $A$  and  $B$  are related by

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B + \vec{\mathbf{r}}_{A,B} \times \vec{\mathbf{p}}_{\text{sys}} \quad (19.5.18)$$

Thus if the momentum of the system is zero, the angular momentum is the same about any point.

$$\vec{\mathbf{L}}_A = \vec{\mathbf{L}}_B, \quad (\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}}). \quad (19.5.19)$$

In particular, the momentum of a system of particles is zero by definition in the center of mass reference frame because in that reference frame  $\vec{\mathbf{p}}_{\text{sys}} = \vec{\mathbf{0}}$ . Hence the angular momentum is the same about any point in the center of mass reference frame.

## 19.7 Angular Momentum and Torque for Fixed Axis Rotation

We have shown that, for fixed axis rotation, the component of torque that causes the angular velocity to change is the rotational analog of Newton's Second Law,

$$\vec{\tau}_S^{\text{ext}} = I_S \vec{\alpha}. \quad (19.5.20)$$

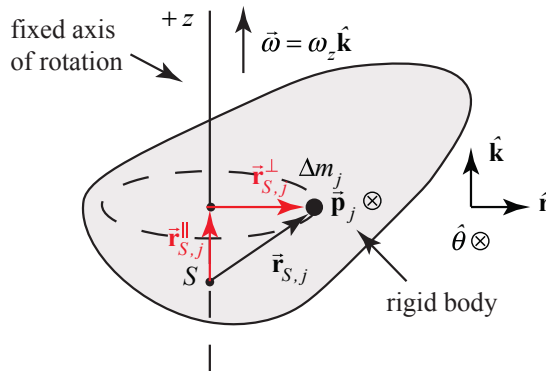
We shall now see that this is a special case of the more general result

$$\vec{\tau}_S^{\text{ext}} = \frac{d}{dt} \vec{L}_S^{\text{sys}}. \quad (19.5.21)$$

Consider a rigid body rotating about a fixed axis passing through the point  $S$  and take the fixed axis of rotation to be the  $z$ -axis. Recall that all the points in the rigid body rotate about the  $z$ -axis with the same angular velocity  $\vec{\omega} = (d\theta/dt)\hat{\mathbf{k}} = \omega_z \hat{\mathbf{k}}$ . In a similar fashion, all points in the rigid body have the same angular acceleration,  $\vec{\alpha} = (d^2\theta/dt^2)\hat{\mathbf{k}} = \alpha_z \hat{\mathbf{k}}$ . Let the point  $S$  lie somewhere along the  $z$ -axis.

As before, the body is divided into individual elements. We calculate the contribution of each element to the angular momentum about the point  $S$ , and then sum over all the elements. The summation will become an integral for a continuous body.

Each individual element has a mass  $\Delta m_j$  and is moving in a circle of radius  $r_{S,j}^\perp$  about the axis of rotation. Let  $\vec{r}_{S,j}$  be the vector from the point  $S$  to the element. The momentum of the element,  $\vec{p}_j$ , is tangent to this circle (Figure 19.16).



**Figure 19.16** Geometry of instantaneous rotation.

The angular momentum of the  $j^{\text{th}}$  element about the point  $S$  is given by  $\vec{L}_{S,j} = \vec{r}_{S,j} \times \vec{p}_j$ . The vector  $\vec{r}_{S,j}$  can be decomposed into parallel and perpendicular components with respect to the axis of rotation  $\vec{r}_{S,j} = \vec{r}_{S,j}^{\parallel} + \vec{r}_{S,j}^{\perp}$  (Figure 19.16), where  $r_{S,j}^{\perp} = |\vec{r}_{S,j}^{\perp}|$  and  $r_{S,j}^{\parallel} = |\vec{r}_{S,j}^{\parallel}|$ . The momentum is given by  $\vec{p}_j = \Delta m_j r_{S,j}^{\perp} \omega_z \hat{\theta}$ . Then the angular momentum about the point  $S$  is

$$\begin{aligned}\vec{\mathbf{L}}_{S,j} &= \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{p}}_j = (r_{S,j}^\perp \hat{\mathbf{r}} + r_{S,j}^\parallel \hat{\mathbf{k}}) \times (\Delta m_j r_{S,j}^\perp \omega_z \hat{\boldsymbol{\theta}}) \\ &= \Delta m_j (r_{S,j}^\perp)^2 \omega_z \hat{\mathbf{k}} - \Delta m_j r_{S,j}^\parallel r_{S,j}^\perp \omega_z \hat{\mathbf{r}}\end{aligned}\quad (19.5.22)$$

In the last expression in Equation (19.5.22), the second term has a direction that is perpendicular to the  $z$ -axis. Therefore the  $z$ -component of the angular momentum about the point  $S$ ,  $(L_{S,j})_z$ , arises entirely from the second term,  $\vec{\mathbf{r}}_{S,j}^\perp \times \vec{\mathbf{p}}_j$ . Therefore the  $z$ -component of the angular momentum about  $S$  is

$$(L_{S,j})_z = \Delta m_j (r_{S,j}^\perp)^2 \omega_z. \quad (19.5.23)$$

The  $z$ -component of the angular momentum of the system about  $S$  is the summation over all the elements,

$$L_{S,z}^{\text{sys}} = \sum_j (L_{S,j})_z = \sum_j \Delta m_j (r_{S,j}^\perp)^2 \omega_z. \quad (19.5.24)$$

For a continuous mass distribution the summation becomes an integral over the body,

$$L_{S,z}^{\text{sys}} = \int_{\text{body}} dm (r_{dm})^2 \omega_z, \quad (19.5.25)$$

where  $r_{dm}$  is the distance from the fixed  $z$ -axis to the infinitesimal element of mass  $dm$ . The moment of inertia of a rigid body about a fixed  $z$ -axis passing through a point  $S$  is given by an integral over the body

$$I_S = \int_{\text{body}} dm (r_{dm})^2. \quad (19.5.26)$$

Thus the  $z$ -component of the angular momentum about  $S$  for a fixed axis that passes through  $S$  in the  $z$ -direction is proportional to the  $z$ -component of the angular velocity,  $\omega_z$ ,

$$L_{S,z}^{\text{sys}} = I_S \omega_z. \quad (19.5.27)$$

For fixed axis rotation, our result that torque about a point is equal to the time derivative of the angular momentum about that point,

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d}{dt} \vec{\mathbf{L}}_S^{\text{sys}}, \quad (19.5.28)$$

can now be resolved in the  $z$ -direction,

$$\tau_{S,z}^{\text{ext}} = \frac{dL_{S,z}^{\text{sys}}}{dt} = \frac{d}{dt} (I_S \omega_z) = I_S \frac{d\omega_z}{dt} = I_S \frac{d^2\theta}{dt^2} = I_S \alpha_z, \quad (19.5.29)$$

in agreement with our earlier result that the  $z$ -component of torque about the point  $S$  is equal to the product of moment of inertia about  $I_S$ , and the  $z$ -component of the angular acceleration,  $\alpha_z$ .

### Example 19.6 Circular Ring

A circular ring of radius  $R$ , and mass  $M$  is rotating about the  $z$ -axis in a plane parallel to but a distance  $h$  above the  $x$ - $y$  plane. The  $z$ -component of the angular velocity is  $\omega_z$  (Figure 19.17). Find the magnitude and the direction of the angular momentum  $\vec{L}_S$  along at any point  $S$  on the central  $z$ -axis.

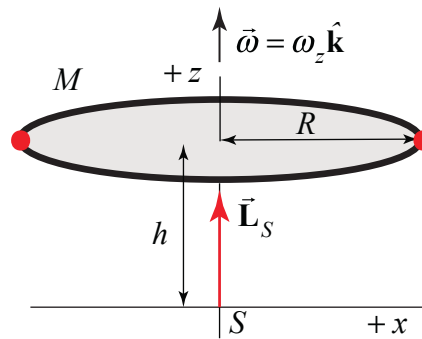


Figure 19.17 Example 19.6

**Solution:** Use the same symmetry argument as we did in Example 19.5. The ring can be thought of as made up of pairs of point like objects on opposite sides of the ring each of mass  $m$  (Figure 19.18).

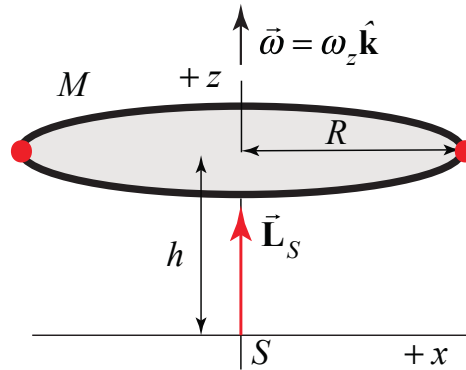


Figure 19.18 Ring as a sum of pairs of symmetrically distributed particles

Each pair has a non-zero  $z$ -component of the angular momentum taken about any point  $S$  along the  $z$ -axis,  $\vec{L}_S^{pair} = \vec{L}_{S,1} + \vec{L}_{S,2} = 2mR^2\omega_z \hat{k} = m^{pair}R^2\omega_z \hat{k}$ . The angular momentum of the ring about the point  $S$  is then the sum over all the pairs

$$\vec{L}_S = \sum_{pairs} m^{pair} R^2 \omega_z \hat{k} = MR^2 \omega_z \hat{k}. \quad (19.5.30)$$



Recall that the moment of inertia of a ring is given by

$$I_S = \int_{\text{body}} dm (r_{dm})^2 = MR^2. \quad (19.5.31)$$

For the symmetric ring, the angular momentum about  $S$  points in the direction of the angular velocity and is equal to

$$\vec{\mathbf{L}}_S = I_S \omega \hat{\mathbf{k}}. \quad (19.5.32)$$

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