16.3 Rotational Kinetic Energy and Moment of Inertia

16.3.1 Rotational Kinetic Energy and Moment of Inertia

We have already defined translational kinetic energy for a point object as $K = (1/2)mv^2$; we now define the rotational kinetic energy for a rigid body about its center of mass.

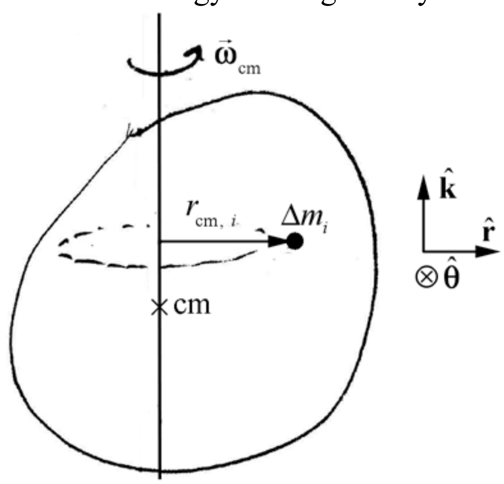


Figure 16.8 Volume element undergoing fixed-axis rotation about the z-axis that passes through the center of mass.

Choose the z-axis to lie along the axis of rotation passing through the center of mass. As in Section 16.2.2, divide the body into volume elements of mass Δm_i (Figure 16.8). Each individual mass element Δm_i undergoes circular motion about the center of mass with z-

component of angular velocity ω_{cm} in a circle of radius $r_{\text{cm},\,i}$. Therefore the velocity of each element is given by $\vec{\mathbf{v}}_{\text{cm},i} = r_{\text{cm},i}\omega_{\text{cm}}\,\hat{\boldsymbol{\theta}}$. The rotational kinetic energy is then

$$K_{\text{cm},\,i} = \frac{1}{2}\Delta m_i\, v_{\text{cm},\,i}^2 = \frac{1}{2}\Delta m_i r_{\text{cm},\,i}^2 \omega_{\text{cm}}^2 . \tag{16.2.1}$$

We now add up the kinetic energy for all the mass elements,

$$\begin{aligned} K_{\text{cm}} &= \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} K_{\text{cm},\,i} = \lim_{\substack{i\to\infty \\ \Delta m_i \to 0}} \sum_{i=1}^{i=N} \left(\sum_i \frac{1}{2}\Delta m_i r_{\text{cm},\,i}^2 \right)\omega_{\text{cm}}^2 \\ &= \left(\frac{1}{2}\int_{\text{body}} dm\, r_{\text{dm}}^2 \right)\omega_{\text{cm}}^2 , \end{aligned} \tag{16.2.2}$$

where dm is an infinitesimal mass element undergoing a circular orbit of radius r_{dm} about the axis passing through the center of mass.

The quantity

$$I_{cm} = \int_{body} dm\, r_{dm}^2 . \tag{16.2.3}$$

is called the ***moment of inertia*** *of the rigid body about a fixed axis passing through the center of mass, and is a physical property of the body. The SI units for moment of inertia are* $\left[\text{kg}\cdot\text{m}^2\right]$.

Thus

$$K_{\text{cm}} = \left(\frac{1}{2}\int_{\text{body}} dm\, r_{\text{dm}}^2 \right)\omega_{\text{cm}}^2 \equiv \frac{1}{2} I_{cm}\omega_{\text{cm}}^2 . \tag{16.2.4}$$

16.3.2 Moment of Inertia of a Rod of Uniform Mass Density

Consider a thin uniform rod of length L and mass m. In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure 16.9. Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the x-axis to lie along the length of the rod, with the positive x-direction to the right, as in the figure.

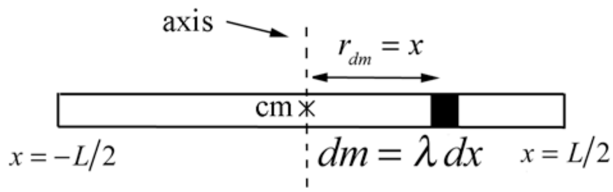


Figure 16.9 Moment of inertia of a uniform rod about center of mass.

Identify an infinitesimal mass element $dm = \lambda\, dx$, located at a displacement x from the center of the rod, where the mass per unit length $\lambda = m/L$ is a constant, as we have assumed the rod to be uniform. When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius $r_{dm} = x$. We add together the contributions from each infinitesimal element as we go from $x = -L/2$ to $x = L/2$. The integral is then

$$
\begin{aligned}
I_{\text{cm}} &= \int_{\text{body}} r_{\text{dm}}^2 \, dm = \lambda \int_{-L/2}^{L/2} (x^2)\, dx = \lambda \left. \frac{x^3}{3} \right|_{-L/2}^{L/2} \\
&= \frac{m}{L}\frac{(L/2)^3}{3} - \frac{m}{L}\frac{(-L/2)^3}{3} = \frac{1}{12} m L^2 .
\end{aligned}
\tag{16.2.5}
$$

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the x- axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the x-y plane if the axis is along the z- axis. The calculation of the moment of inertia in these cases would be more complicated.)

Example 16.2 Moment of Inertia of a Uniform Disc

A thin uniform disc of mass M and radius R is mounted on an axle passing through the center of the disc, perpendicular to the plane of the disc. Calculate the moment of inertia about an axis that passes perpendicular to the disc through the center of mass of the disc

Solution: As a starting point, consider the contribution to the moment of inertia from the mass element dm show in Figure 16.10. Let r denote the distance form the center of mass of the disc to the mass element.

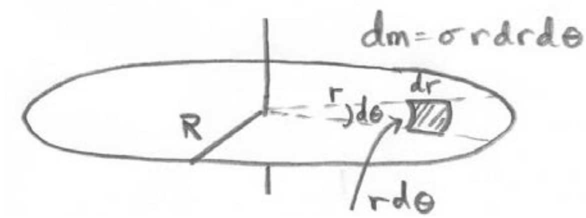


Figure 16.10 Infinitesimal mass element and coordinate system for disc.

Choose cylindrical coordinates with the coordinates (r,θ) in the plane and the z-axis perpendicular to the plane. The area element

$$da = r\,dr\,d\theta \tag{16.2.6}$$

may be thought of as the product of arc length $r\,d\theta$ and the radial width dr. Since the disc is uniform, the mass per unit area is a constant,

$$\sigma = \frac{dm}{da} = \frac{m_{\text{total}}}{\text{Area}} = \frac{M}{\pi R^2}\,. \tag{16.2.7}$$

Therefore the mass in the infinitesimal area element as given in Equation (16.2.6), a distance r from the axis of rotation, is given by

$$dm = \sigma\, r\,dr\,d\theta = \frac{M}{\pi R^2} r\,dr\,d\theta\,. \tag{16.2.8}$$

When the disc rotates, the mass element traces out a circle of radius $r_{dm} = r$; that is, the distance from the center is the perpendicular distance from the axis of rotation. The moment of inertia integral is now an integral in two dimensions; the angle θ varies from $\theta = 0$ to $\theta = 2\pi$, and the radial coordinate r varies from $r = 0$ to $r = R$. Thus the limits of the integral are

$$I_{\text{cm}} = \int_{\text{body}} r_{dm}^2\, dm = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\int_{\theta=0}^{\theta=2\pi} r^3\, d\theta\, dr\,. \tag{16.2.9}$$

The integral can now be explicitly calculated by first integrating the θ-coordinate

$$I_{\text{cm}} = \frac{M}{\pi R^2}\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) r^3 dr = \frac{M}{\pi R^2}\int_{r=0}^{r=R} 2\pi r^3 dr = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr \tag{16.2.10}$$

and then integrating the r-coordinate,

$$I_{\text{cm}} = \frac{2M}{R^2}\int_{r=0}^{r=R} r^3 dr = \frac{2M}{R^2}\frac{r^4}{4}\Bigg|_{r=0}^{r=R} = \frac{2M}{R^2}\frac{R^4}{4} = \frac{1}{2}MR^2\,. \tag{16.2.11}$$

Remark: Instead of taking the area element as a small patch $da = r\,dr\,d\theta$, choose a ring of radius r and width dr. Then the area of this ring is given by

$$da_{\text{ring}} = \pi(r+dr)^2 - \pi r^2 = \pi r^2 + 2\pi r\,dr + \pi(dr)^2 - \pi r^2 = 2\pi r\,dr + \pi(dr)^2\,. \tag{16.2.12}$$

In the limit that $dr \to 0$, the term proportional to $(dr)^2$ can be ignored and the area is $da = 2\pi r dr$. This equivalent to first integrating the $d\theta$ variable

$$da_{\text{ring}} = r\,dr\left(\int_{\theta=0}^{\theta=2\pi} d\theta\right) = 2\pi r\,dr\,. \tag{16.2.13}$$

Then the mass element is

$$dm_{\text{ring}} = \sigma da_{\text{ring}} = \frac{M}{\pi R^2} 2\pi r\,dr\,. \tag{16.2.14}$$

The moment of inertia integral is just an integral in the variable r,

$$I_{\text{cm}} = \int_{\text{body}} (r_{\perp})^2\,dm = \frac{2\pi M}{\pi R^2}\int_{r=0}^{r=R} r^3 dr = \frac{1}{2}MR^2\,. \tag{16.2.15}$$

16.3.3 Parallel Axis Theorem

Consider a rigid body of mass m undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is I_{cm}. The second axis passes through some other point S in the body. Let $d_{S,\text{cm}}$ denote the perpendicular distance between the two parallel axes (Figure 16.11).

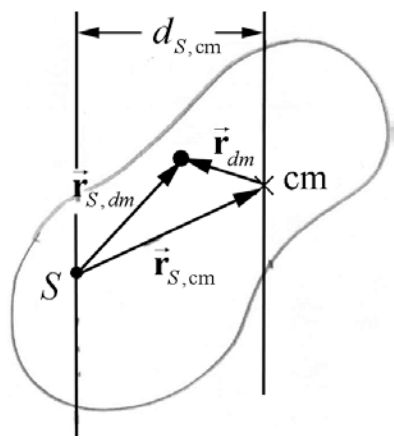


Figure 16.11 Geometry of the parallel axis theorem.

Then the moment of inertia I_S about an axis passing through a point S is related to I_{cm} by

$$I_S = I_{\text{cm}} + m\,d_{S,\text{cm}}^2\,. \tag{16.2.16}$$

16.3.4 Parallel Axis Theorem Applied to a Uniform Rod

Let point S be the left end of the rod of Figure 16.9. Then the distance from the center of mass to the end of the rod is $d_{S,\mathrm{cm}} = L/2$. The moment of inertia $I_S = I_{\mathrm{end}}$ about an axis passing through the endpoint is related to the moment of inertia about an axis passing through the center of mass, $I_{\mathrm{cm}} = (1/12)\,m\,L^2$, according to Equation (16.2.16),

$$I_S = \frac{1}{12} m\,L^2 + \frac{1}{4} m\,L^2 = \frac{1}{3} m\,L^2 . \qquad (16.2.17)$$

In this case it's easy and useful to check by direct calculation. Use Equation (16.2.5) but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$,

$$\begin{aligned} I_{\mathrm{end}} &= \int_{\text{body}} r_\perp^{\,2}\, dm = \lambda \int_0^L x'^2\, dx' \\ &= \lambda \left.\frac{x'^3}{3}\right|_0^L = \frac{m}{L}\frac{(L)^3}{3} - \frac{m}{L}\frac{(0)^3}{3} = \frac{1}{3} m\,L^2 . \end{aligned} \qquad (16.2.18)$$

Example 16.3 Rotational Kinetic Energy of Disk

A disk with mass M and radius R is spinning with angular speed ω about an axis that passes through the rim of the disk perpendicular to its plane. The moment of inertia about cm is $I_{cm} = (1/2)mR^2$. What is the kinetic energy of the disk?

Solution: The parallel axis theorem states the moment of inertia about an axis passing perpendicular to the plane of the disc and passing through a point on the edge of the disc is equal to

$$I_{edge} = I_{cm} + mR^2 . \qquad (16.2.19)$$

The moment of inertia about an axis passing perpendicular to the plane of the disc and passing through the center of mass of the disc is equal to $I_{cm} = (1/2)mR^2$. Therefore

$$I_{edge} = (3/2)mR^2 . \qquad (16.2.20)$$

The kinetic energy is then

$$K = (1/2) I_{edge}\,\omega^2 = (3/4)mR^2\omega^2 . \qquad (16.2.21)$$

16.4 Conservation of Energy for Fixed Axis Rotation

Consider a closed system ($\Delta E_{system} = 0$) under action of only conservative internal forces. Then the change in the mechanical energy of the system is zero

$$\Delta E_m = \Delta U + \Delta K = (U_f + K_f) - (U_i + K_i) = 0 . \qquad (16.3.1)$$

For fixed axis rotation with a component of angular velocity ω about the fixed axis, the change in kinetic energy is given by

$$\Delta K \equiv K_f - K_i = \frac{1}{2} I_S \omega_f^2 - \frac{1}{2} I_S \omega_i^2 , \qquad (16.3.2)$$

where S is a point that lies on the fixed axis. Then conservation of energy implies that

$$U_f + \frac{1}{2} I_S \omega_f^2 = U_i + \frac{1}{2} I_S \omega_i^2 \qquad (16.3.3)$$

Example 16.4 Energy and Pulley System

A wheel in the shape of a uniform disk of radius R and mass m_p is mounted on a frictionless horizontal axis. The wheel has moment of inertia about the center of mass $I_\mathrm{cm} = (1/2) m_\mathrm{p} R^2$. A massless cord is wrapped around the wheel and one end of the cord is attached to an object of mass m_2 that can slide up or down a frictionless inclined plane. The other end of the cord is attached to a second object of mass m_1 that hangs over the edge of the inclined plane. The plane is inclined from the horizontal by an angle θ (Figure 16.12). Once the objects are released from rest, the cord moves without slipping around the disk. Calculate the speed of block 2 as a function of distance that it moves down the inclined plane using energy techniques. Assume there are no energy losses due to friction and that the rope does not slip around the pulley

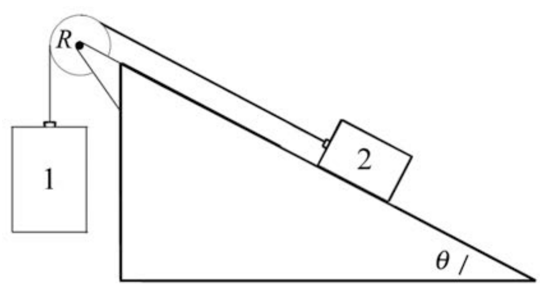


Figure 16.12 Pulley and blocks

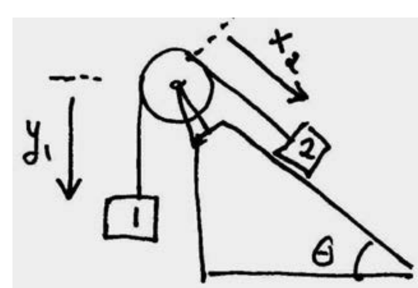
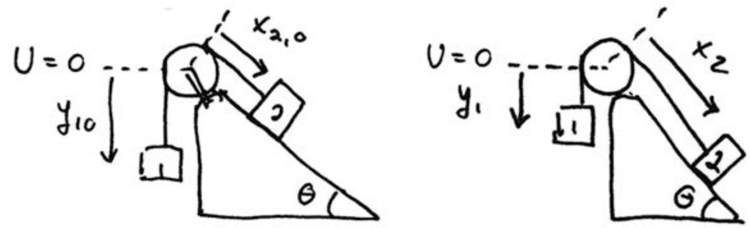


Figure 16.13 Coordinate system for pulley and blocks

Solution: Define a coordinate system as shown in Figure 16.13. Choose the zero for the gravitational potential energy at a height equal to the center of the pulley. In Figure 16.14 illustrates the energy diagrams for the initial state and a dynamic state at an arbitrary time when the blocks are sliding.



initial state dynamic state

Figure 16.14 Energy diagrams for initial state and dynamic state at arbitrary time

Then the initial mechanical energy is

$$E_i = U_i = -m_1 g y_{1,i} - m_2 g x_{2,i} \sin\theta \,. \tag{16.3.4}$$

The mechanical energy, when block 2 has moved a distance

$$d = x_2 - x_{2,i} \tag{16.3.5}$$

is given by

$$E = U + K = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2} m_1 v_1^{\,2} + \frac{1}{2} m_2 v_2^{\,2} + \frac{1}{2} I_P \omega^2 \,. \tag{16.3.6}$$

The rope connects the two blocks, and so the blocks move at the same speed

$$v \equiv v_1 = v_2 \,. \tag{16.3.7}$$

The rope does not slip on the pulley; therefore as the rope moves around the pulley the tangential speed of the rope is equal to the speed of the blocks

$$v_{\tan} = R\omega = v \,. \tag{16.3.8}$$

Eq. (16.3.6) can now be simplified

$$E = U + K = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2} \right) v^2 \,. \tag{16.3.9}$$

Because we have assumed that there is no loss of mechanical energy, we can set $E_i = E$ and find that

$$-m_1 g y_{1,i} - m_2 g x_{2,i}\sin\theta = -m_1 g y_1 - m_2 g x_2 \sin\theta + \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2}\right)v^2, \quad (16.3.10)$$

which simplifies to

$$-m_1 g(y_{1,0} - y_1) + m_2 g(x_2 - x_{2,0})\sin\theta = \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2}\right)v^2. \quad (16.3.11)$$

We finally note that the movement of block 1 and block 2 are constrained by the relationship

$$d = x_2 - x_{2,i} = y_{1,i} - y_1. \quad (16.3.12)$$

Then Eq. (16.3.11) becomes

$$gd(-m_1 + m_2\sin\theta) = \frac{1}{2}\left(m_1 + m_2 + \frac{I_P}{R^2}\right)v^2. \quad (16.3.13)$$

We can now solve for the speed as a function of distance $d = x_2 - x_{2,i}$ that block 2 has traveled down the incline plane

$$v = \sqrt{\frac{2gd(-m_1 + m_2\sin\theta)}{\left(m_1 + m_2 + (I_P / R^2)\right)}}. \quad (16.3.14)$$

If we assume that the moment of inertial of the pulley is $I_{\mathrm{cm}} = (1/2)m_{\mathrm{p}}R^2$, then the speed becomes

$$v = \sqrt{\frac{2gd(-m_1 + m_2\sin\theta)}{\left(m_1 + m_2 + (1/2)m_P\right)}}. \quad (16.3.15)$$

Example 16.5 Physical Pendulum

A physical pendulum consists of a uniform rod of mass m_1 pivoted at one end about the point S. The rod has length l_1 and moment of inertia I_1 about the pivot point. A disc of mass m_2 and radius r_2 with moment of inertia I_{cm} about its center of mass is rigidly attached a distance l_2 from the pivot point. The pendulum is initially displaced to an angle θ_i and then released from rest. (a) What is the moment of inertia of the physical pendulum about the pivot point S? (b) How far from the pivot point is the center of mass of the system? (c) What is the angular speed of the pendulum when the pendulum is at the bottom of its swing?

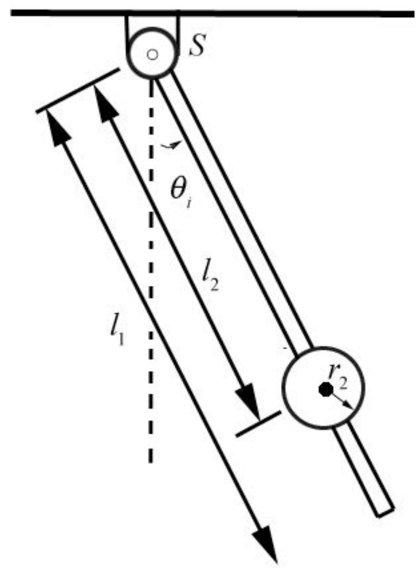


Figure 16.15 Rod and with fixed disc pivoted about the point S

Solution: a) The moment of inertia about the pivot point is the sum of the moment of inertia of the rod, given as I_1, and the moment of inertia of the disc about the pivot point. The moment of inertia of the disc about the pivot point is found from the parallel axis theorem,

$$I_{\text{disc}} = I_{\text{cm}} + m_2\, l_2^2 . \tag{16.3.16}$$

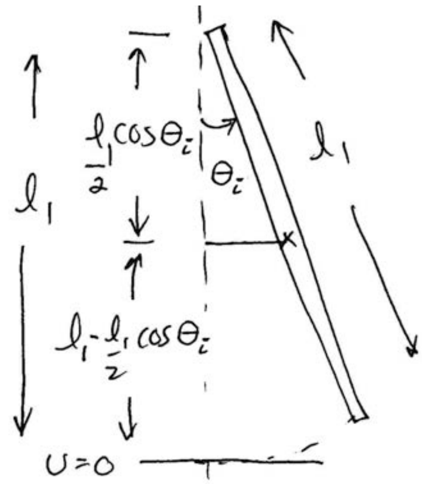
The moment of inertia of the system consisting of the rod and disc about the pivot point S is then

$$I_S = I_1 + I_{\text{disc}} = I_1 + I_{\text{cm}} + m_2\, l_2^2 . \tag{16.3.17}$$

The center of mass of the system is located a distance from the pivot point

$$l_{\text{cm}} = \frac{m_1(l_1 / 2) + m_2\, l_2}{m_1 + m_2} . \tag{16.3.18}$$

b) We can use conservation of mechanical energy, to find the angular speed of the pendulum at the bottom of its swing. Take the zero point of gravitational potential energy to be the point where the bottom of the rod is at its lowest point, that is, $\theta = 0$. The initial state energy diagram for the rod is shown in Figure 16.16a and the initial state energy diagram for the disc is shown in Figure 16.16b.



(a) (b)

Figure 16.16 (a) Initial state energy diagram for rod (b) Initial state energy diagram for disc

The initial mechanical energy is then

$$E_i = U_i = m_1 g(l_1 - \frac{l_1}{2}\cos\theta_i) + m_2 g(l_1 - l_2\cos\theta_i)\,, \tag{16.3.19}$$

At the bottom of the swing, $\theta_f = 0$, and the system has angular velocity ω_f. The mechanical energy at the bottom of the swing is

$$E_f = U_f + K_f = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.20}$$

with I_S as found in Equation (16.3.17). There are no non-conservative forces acting, so the mechanical energy is constant therefore equating the expressions in (16.3.19) and (16.3.20) we get that

$$m_1 g(l_1 - \frac{l_1}{2}\cos\theta_i) + m_2 g(l_1 - l_2\cos\theta_i) = m_1 g\frac{l_1}{2} + m_2 g(l_1 - l_2) + \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.21}$$

This simplifies to

$$\left(\frac{m_1 l_1}{2} + m_2 l_2\right)g(1 - \cos\theta_i) = \frac{1}{2}I_S\omega_f^2\,, \tag{16.3.22}$$

We now solve for ω_f (taking the positive square root to insure that we are calculating angular speed)

$$\omega_f = \sqrt{\frac{2\left(\frac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_i)}{I_S}}\,, \qquad (16.3.23)$$

Finally we substitute in Eq.(16.3.17) in to Eq. (16.3.23) and find

$$\omega_f = \sqrt{\frac{2\left(\frac{m_1 l_1}{2} + m_2 l_2\right) g(1-\cos\theta_i)}{I_1 + I_{\mathrm{cm}} + m_2 l_2^2}}\,. \qquad (16.3.24)$$

Note that we can rewrite Eq. (16.3.22), using Eq. (16.3.18) for the distance between the center of mass and the pivot point, to get

$$(m_1 + m_2) l_{cm} g(1-\cos\theta_i) = \frac{1}{2} I_S \omega_f^2\,, \qquad (16.3.25)$$

We can interpret this equation as follows. Treat the system as a point particle of mass $m_1 + m_2$ located at the center of mass l_{cm}. Take the zero point of gravitational potential energy to be the point where the center of mass is at its lowest point, that is, $\theta = 0$. Then

$$E_i = (m_1 + m_2) l_{cm} g(1-\cos\theta_i)\,, \qquad (16.3.26)$$

$$E_f = \frac{1}{2} I_S \omega_f^2\,. \qquad (16.3.27)$$

Thus conservation of energy reproduces Eq. (16.3.25).