13.11 Work-Kinetic Energy Theorem in Three Dimensions

Recall our mathematical result that for one-dimensional motion

$$m\int_i^f a_x\,dx = m\int_i^f \frac{dv_x}{dt}\,dx = m\int_i^f dv_x\,\frac{dx}{dt} = m\int_i^f v_x\,dv_x = \frac{1}{2}mv_{x,f}^2 - \frac{1}{2}mv_{x,i}^2\,. \quad (13.11.1)$$

Using Newton's Second Law in the form $F_x = m\,a_x$, we concluded that

$$\int_i^f F_x\,dx = \frac{1}{2}mv_{x,f}^2 - \frac{1}{2}mv_{x,i}^2\,. \quad (13.11.2)$$

Eq. (13.11.2) generalizes to the y- and z-directions:

$$\int_i^f F_y\,dy = \frac{1}{2}mv_{y,f}^2 - \frac{1}{2}mv_{y,i}^2\,, \quad (13.11.3)$$

$$\int_i^f F_z\,dz = \frac{1}{2}mv_{z,f}^2 - \frac{1}{2}mv_{z,i}^2\,. \quad (13.11.4)$$

Adding Eqs. (13.11.2), (13.11.3), and (13.11.4) yields

$$\int_i^f (F_x\,dx + F_y\,dy + F_z\,dz) = \frac{1}{2}m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2}m(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2)\,. \quad (13.11.5)$$

Recall (Eq. (13.8.24)) that the left hand side of Eq. (13.11.5) is the work done by the force $\vec{\mathbf{F}}$ on the object

$$W = \int_i^f dW = \int_i^f (F_x\,dx + F_y\,dy + F_z\,dz) = \int_i^f \vec{\mathbf{F}}\cdot d\vec{\mathbf{r}} \quad (13.11.6)$$

The right hand side of Eq. (13.11.5) is the change in kinetic energy of the object

$$\Delta K \equiv K_f - K_i = \frac{1}{2}m{v_f}^2 - \frac{1}{2}m{v_0}^2 = \frac{1}{2}m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2}m(v_{x,i}^2 + v_{y,i}^2 + v_{z,i}^2)\,. \quad (13.11.7)$$

Therefore Eq. (13.11.5) is the three dimensional generalization of the work-kinetic energy theorem

$$\int_i^f \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = K_f - K_i. \qquad (13.11.8)$$

When the work done on an object is positive, the object will increase its speed, and negative work done on an object causes a decrease in speed. When the work done is zero, the object will maintain a constant speed.

13.11.1 Instantaneous Power Applied by a Non-Constant Force for Three Dimensional Motion

Recall that for one-dimensional motion, the *instantaneous power* at time t is defined to be the limit of the average power as the time interval $[t, t + \Delta t]$ approaches zero,

$$P(t) = F_x^a(t) v_x(t) \,. \qquad (13.11.9)$$

A more general result for the instantaneous power is found by using the expression for dW as given in Equation (13.8.23),

$$P = \frac{dW}{dt} = \frac{\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}}{dt} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} \,. \qquad (13.11.10)$$

The time rate of change of the kinetic energy for a body of mass m is equal to the power,

$$\frac{dK}{dt} = \frac{1}{2} m \frac{d}{dt}\left(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} \right) = m \frac{d\vec{\mathbf{v}}}{dt} \cdot \vec{\mathbf{v}} = m \vec{\mathbf{a}} \cdot \vec{\mathbf{v}} = \vec{\mathbf{F}} \cdot \vec{\mathbf{v}} = P \,. \qquad (13.11.11)$$

where the we used Eq. (13.8.9), Newton's Second Law and Eq. (13.11.10).

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