8.012 Physics I: Classical Mechanics
Fall 2008

Problem 1 of 5

Short answer questions. (10 points)

Note: DO NOT spend too much time on this problem!
It is worth fewer points than the other 4 problems.

For each question, circle the correct answer:

(a) (2 points) A bee and a train collide. Compared to the magnitude of the force experienced by the bee, the magnitude of the force experienced by the train is

smaller   larger   (equal)   dependent on other details

(from Newton's third law)

(b) (2 points) A point mass m is placed inside a thin spherical shell of mass M and radius R, at a distance $R/2$ from the center. Let $\hat{r}$ be a radial unit vector pointing from the point mass away from the center of the shell. The gravitational force exerted on the point mass by the spherical shell is

$-\frac{4GMm}{R^2}\hat{r}$ $-\frac{GMm}{R^2}\hat{r}$ (zero) $+\frac{GMm}{R^2}\hat{r}$ $+\frac{4GMm}{R^2}\hat{r}$

(c) (2 points) A mass M suspended from the ceiling by a spring (with spring constant k) undergoes simple harmonic motion with an oscillation period P. If the mass is doubled, then the new oscillation period is

$P/2$ $P/\sqrt{2}$ P $(P\sqrt{2})$ $2P$

$\omega = \sqrt{\frac{k}{m}}$, $P = \frac{2\pi}{\omega}$

(d) (2 points) A block of mass M slides down a inclined plane with friction coefficient μ. The surface area of the block in contact with the inclined plane is A, and the frictional force on the block is f. If we use a different block (made of the same material) with contact area $2A$ but the same mass M, then the frictional force on the second block will be approximately

$f/4$ $f/2$ (f) $2f$ $4f$

$f = \mu N$ depends on weight but not on surface area

(e) (2 points) When a mass moves through air or fluid at low velocity, it experiences a viscous drag force. (This is different than the turbulent drag felt at high velocity.) For a low velocity v, the viscous drag force is approximately proportional to

(v) v^2 v^3 v^4 v^5

**Masses on an accelerating pulley.** (20 points)

Masses M_1 and M_2 are connected by a massless, inextensible string over a frictionless pulley of negligible mass. The pulley accelerates upward with constant acceleration A due to an external force. Gravity is directed downward.

(a) (5 points) Draw the force diagrams for M_1 and M_2.

(b) (15 points) Find the accelerations of M_1 and M_2 in the inertial x-y reference frame shown.
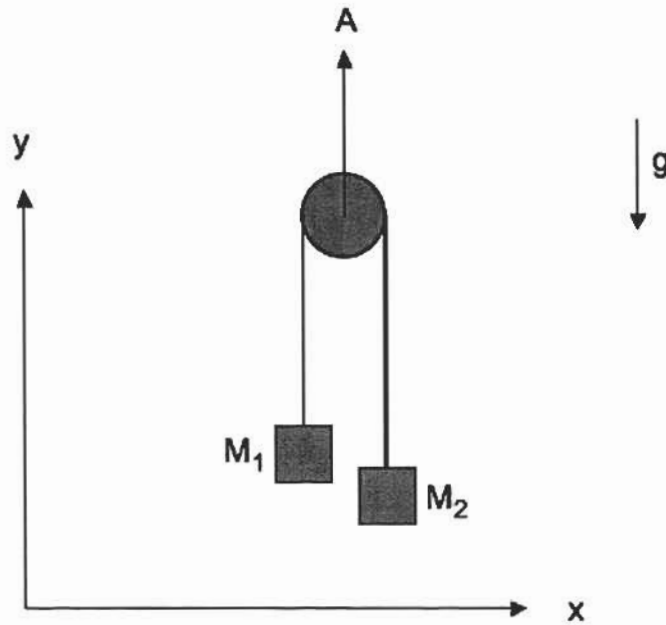
(5 points)

(a)
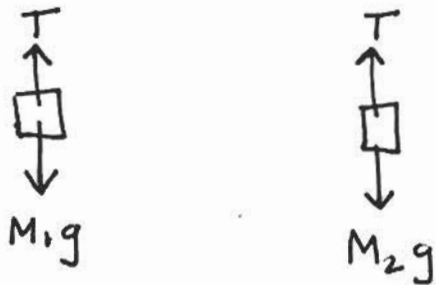
Tension acting on both masses is identical since the rope is massless.

② (cont.)
___

(b) The equations of motion for the masses are

① $\boxed{T - M_1 g = M_1 \ddot{y}_1}$

② $\boxed{T - M_2 g = M_2 \ddot{y}_2}$ equations of motion

There are also 2 constraints:

- The pulley has constant acceleration: $\ddot{y}_p = A$
- The rope has fixed length:

$$\underbrace{(y_p - y_1)}_{\text{left side}} + \underbrace{\pi R}_{\substack{\text{around} \\ \text{pulley}}} + \underbrace{(y_p - y_2)}_{\text{right side}} = L$$

Differentiating twice,

$$2\ddot{y}_p - \ddot{y}_1 - \ddot{y}_2 = 0 \implies 2A - \ddot{y}_1 - \ddot{y}_2 = 0$$

$$\implies \boxed{\ddot{y}_2 = 2A - \ddot{y}_1} \quad ③ \quad \text{constraint}$$

Now we can solve for the accelerations. From equations ② + ③

$$T - M_2 g = M_2 (2A - \ddot{y}_1) \implies T = M_2 (2A + g - \ddot{y}_1)$$

Substituting into equation ①,

$$M_2 (2A + g - \ddot{y}_1) - M_1 g = M_1 \ddot{y}_1$$

$$\implies \boxed{\ddot{y}_1 = \frac{-M_1 g + M_2 (2A + g)}{M_1 + M_2}}$$

and from equation ③,

$$\boxed{\ddot{y}_2 = \frac{-M_2 g + M_1 (2A + g)}{M_1 + M_2}}$$

Your answers should make physical sense
(correct dimensional units, symmetry of exchanging $M_1 \leftrightarrow M_2$,
limiting behavior for $M_1 \to 0$ or $M_2 \to 0$, etc.)

Ant on a spinning turntable. (20 points)

An ant of mass M is walking along a radial line painted on a horizontal disk that is rotating with constant angular velocity Ω. The ant is moving toward the center of the disk with constant radial speed v along the painted line, having started at $r = r_0$ at time $t = 0$. You may assume that gravity is directed downward and that the friction is high enough that the ant never slips. Find the *magnitude* of the force on the ant at a time $t > 0$ before it reaches the center of the disk.

top view   side view
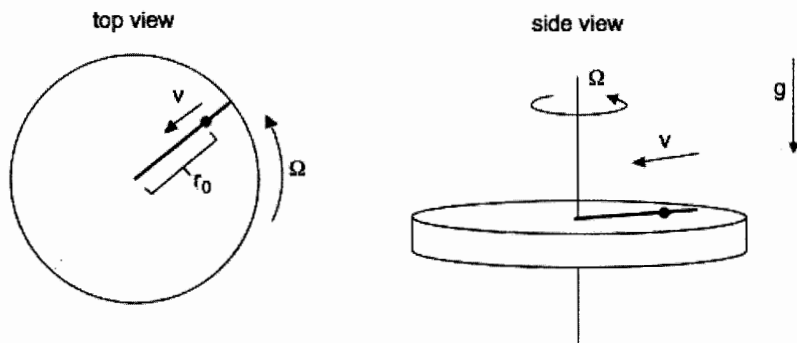
In polar coordinates,

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

For this problem,

$$r = r_0 - vt \qquad \ddot{r} = 0$$
$$\dot{r} = -v \qquad \ddot{\theta} = 0$$
$$\dot{\theta} = \Omega$$

So,

$$\ddot{\vec{r}} = -r\dot{\theta}^2\,\hat{r} + 2\dot{r}\dot{\theta}\,\hat{\theta} = -\Omega^2(r_0 - vt)\,\hat{r} - 2v\Omega\,\hat{\theta}$$

$$|\vec{a}| = |\ddot{\vec{r}}| = \left[\Omega^4(r_0 - vt)^2 + 4v^2\Omega^2\right]^{1/2} = \Omega\left[\Omega^2(r_0 - vt)^2 + 4v^2\right]^{1/2}$$

$$\boxed{|\vec{F}| = m|\vec{a}| = m\Omega\left[\Omega^2(r_0 - vt)^2 + 4v^2\right]^{1/2}}$$

4. Problem 4 of 5

Dog on a springy leash. (25 points)

A dog with mass M has its leash attached to one end of a spring which runs without friction along a horizontal overhead rod. The other end of the spring is fixed to a wall. The spring constant is k. The leash is stiff, massless and inextensible, and it maintains a constant angle θ with the overhead rod, even when the dog moves. There is friction with coefficient μ between the dog and the ground. Gravity is directed downward.

(a) (10 points) Draw force diagrams for the dog and the spring.

(b) (15 points) What is the maximum distance that the dog can stretch the spring beyond its rest length?

(a)

Dog

Mg

Note that friction acts to the right.
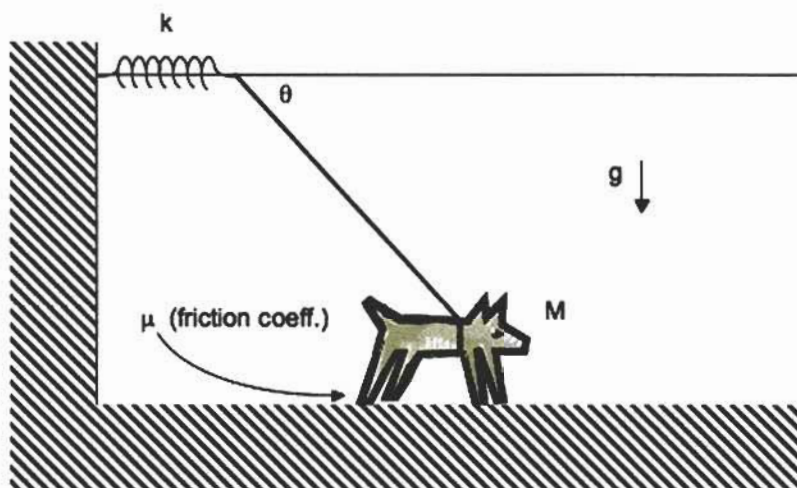
Spring

$(M_{spring} g)$

(10 points)
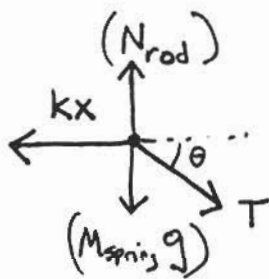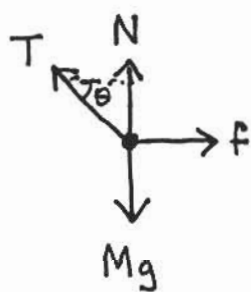
spring has no vertical motion, by constraint.

④ (cont.) At maximum stretch, force balance ⟹ zero acceleration.

(b) For the dog,
$$N + T \sin \theta - Mg = 0 \qquad \text{(vertical)}$$
$$f - T \cos \theta = 0 \qquad \text{(horizontal)}$$

For maximum extension,
$$f = \mu N$$

For spring,
$$T \cos \theta - kx = 0 \qquad \text{(horizontal)}$$

We have 4 unknowns (N, T, f, x) and 4 equations.

Solve for T:
$$T \cos \theta = kx \implies T = \frac{kx}{\cos \theta}$$

Substitute for f and solve for N
$$\mu N - T \cos \theta = 0$$
$$N = \frac{T \cos \theta}{\mu} = \frac{kx}{\mu}$$

So,
$$\frac{kx}{\mu} + \frac{kx \sin \theta}{\cos \theta} = Mg$$
$$kx (1 + \mu \tan \theta) = \mu Mg$$

$$\boxed{x = \frac{\mu Mg}{k(1 + \mu \tan \theta)}}$$ maximum stretch of spring
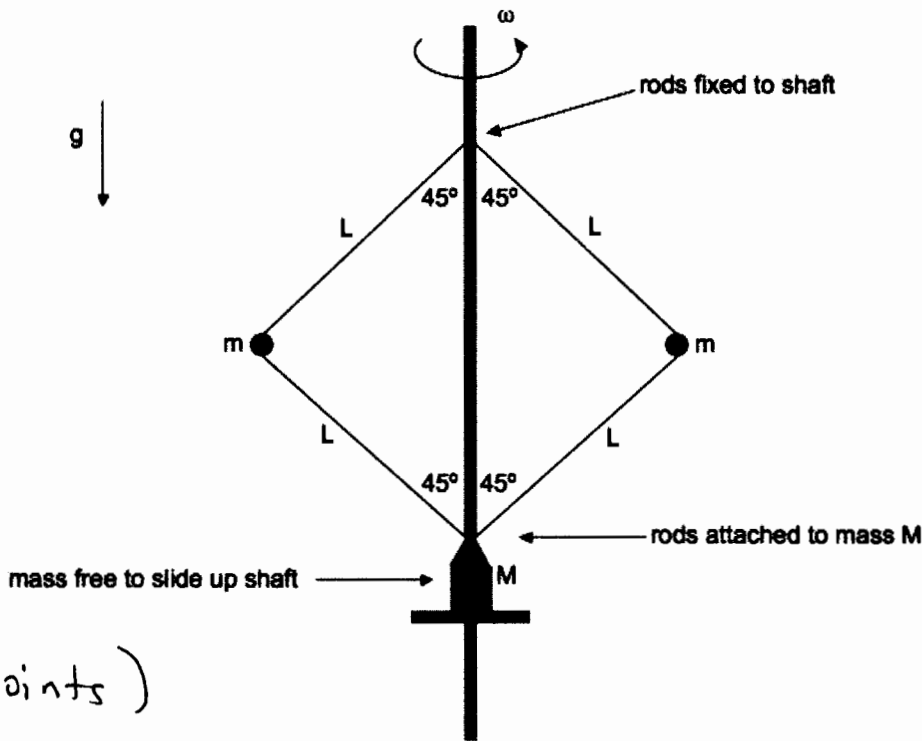
5. Problem 5 of 5
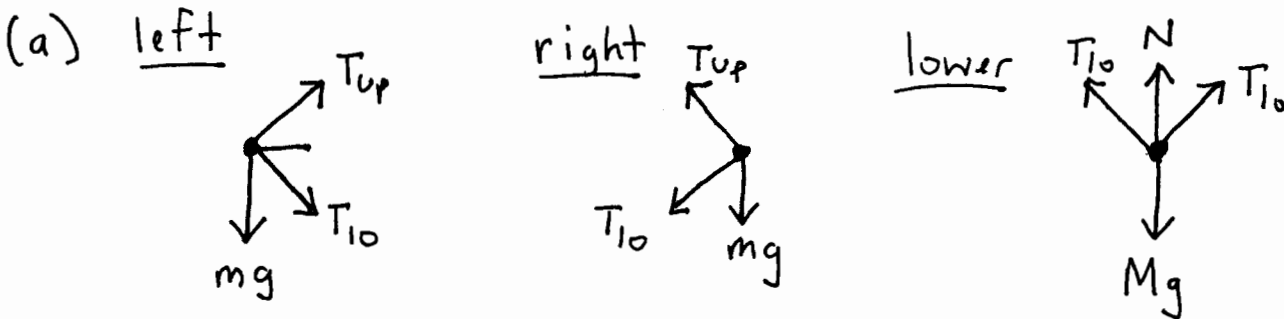
Spinning masses. (25 points)

Two equal masses m are attached by hinges and massless rods of length L to a rotating shaft and to a mass M which rotates with the shaft but is free to slide up it without friction. The upper rods are hinged to a fixed point on the shaft. At low angular velocity ω, the mass M sits on a seat attached to the shaft and the rods make a 45° angle with the shaft, as shown. However, if ω exceeds a particular value ω_0, then the mass M is no longer in contact with its seat but instead begins to slide up the shaft (thus increasing the angle between the rods and the shaft). Gravity acts downwards.

(a) (8 points) Draw force diagrams for the three masses.

(b) (8 points) Write down the equations of motion for each of the masses at low angular velocity ω, when the mass M sits on the seat.

(c) (9 points) Calculate the critical angular velocity ω_0 at which the mass M just begins to rise off the seat and up the shaft.

Hint: The tension in the upper rods is not equal to the tension in the lower rods.

ω

rods fixed to shaft

g

45° 45°

L L

m m

L L

45° 45°

rods attached to mass M

mass free to slide up shaft $\longrightarrow$ M

$\left(8 \text{ points} \right)$

(a) <u>left</u>

T_{up}

T_{lo}

mg

<u>right</u> T_{up}

T_{lo}

mg

<u>lower</u> T_{lo} N T_{lo}

Mg

⑤ (cont.)

(b) Note that $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

The small masses execute uniform circular motion with
radius $R = \frac{L}{\sqrt{2}}$ and centripetal acceleration $\omega^2 R$ inward.

So, for left and right masses:

inward: $\frac{T_{up}}{\sqrt{2}} + \frac{T_{lo}}{\sqrt{2}} = \frac{m\omega^2 L}{\sqrt{2}}$ $\Rightarrow$ $\boxed{T_{up} + T_{lo} = m\omega^2 L}$ ①

vertical: $\frac{T_{up}}{\sqrt{2}} - \frac{T_{lo}}{\sqrt{2}} - mg = 0$ $\Rightarrow$ $\boxed{T_{up} - T_{lo} = mg\sqrt{2}}$ ②

For the lower mass,

vertical: $\frac{2T_{lo}}{\sqrt{2}} + N - Mg = 0$ $\Rightarrow$ $\boxed{N + T_{lo}\sqrt{2} = Mg}$ ③

$\left[\text{horizontal:} \quad \frac{T_{lo}}{\sqrt{2}} - \frac{T_{lo}}{\sqrt{2}} = 0 \right]$

(8 points)

(c) At the critical angular velocity ω_0, we must have $N = 0$.
So, from equation ③,

$$T_{lo} = \frac{Mg}{\sqrt{2}}$$

and from equation ②

$$T_{up} = T_{lo} + mg\sqrt{2} = \frac{Mg}{\sqrt{2}} + mg\sqrt{2}$$

and from equation ①

$$mL\omega_0^2 = T_{up} + T_{lo} = \frac{2Mg}{\sqrt{2}} + mg\sqrt{2} = g\sqrt{2}(M+m)$$

$\Rightarrow$ $\boxed{\omega_0 = \left[\frac{g\sqrt{2}(M+m)}{mL} \right]^{1/2}}$

(9 points)

Useful Relations

- Velocity in polar coordinates

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

- Acceleration in polar coordinates

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\boldsymbol{\theta}}$$