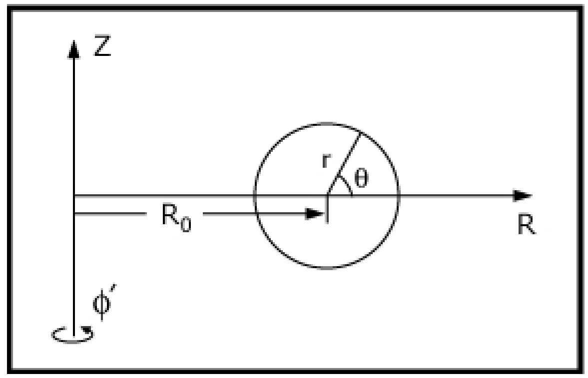
Lecture 6: The Safety Factor and the Ohmic Tokamak

Calculate $q(\psi)$ in an Axisymmetric Torus

1. Introduce toroidal coordinates

   $R = R_0 + r\cos\theta$

   $Z = r\sin\theta$



2. Calculate the field line trajectory by means of the tangent curve

   $$\frac{dr}{B_r} = \frac{rd\theta}{B_\theta} = \frac{Rd\phi}{B_\phi}$$

3. As the magnetic line wraps exactly once around the poloidal cross section (i.e. $\Delta\theta = 2\pi$) it travels along an angle $\Delta\phi$ in the toroidal direction given by

   $$\Delta\phi = \int_0^{\Delta\phi} d\phi = \int_0^{\Delta\theta=2\pi} \frac{d\phi}{d\theta} d\theta$$

   $$= \int_0^{2\pi} d\theta \frac{rB_\phi}{RB_\theta}$$

4. The rotational transform is the average $\Delta\theta$ per single transit in the toroidal direction $\Delta\phi = 2\pi$.

5. Because of symmetry, the pattern of field line motion repeats once $\Delta\theta = 2\pi$. Thus, we simply have to take proportions:

   average $\Delta\theta$ per toroidal transit

   $$\frac{\Delta\theta}{\Delta\phi} : \frac{2\pi}{\Delta\phi} = \frac{\iota}{2\pi}$$

   one toroidal transit $\Delta\phi = 2\pi$

6. Safety factor:

$$q = \left(\frac{\iota}{2\pi}\right)^{-1} = \frac{\Delta\phi}{2\pi}$$

$$q(\psi) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{rB_\phi}{RB_p}\right)_s d\theta = \frac{F}{2\pi}\int\frac{dl_p}{R|\nabla\psi|}$$

7. This integral is difficult to calculate in general because the integrand must be evaluated on the flux surface

   a. $\psi(r,\theta) = \psi_0 = \text{const.} \to r = r(\psi_0,\theta)$ projection of a field line trajectory

   b. Note that on a flux surface

   $$B_\phi = B_\phi\left[r(\theta, \psi_0), \theta\right]$$

   $$B_\theta = B_\theta\left[r(\theta, \psi_0), \theta\right]$$

   $$B_r = B_r\left[r(\theta, \psi_0), \theta\right]$$

   c. We find $r(\theta, \psi_0)$ as follows

   $$\frac{dr}{d\theta} = \frac{rB_r(r,\theta)}{B_\theta(r,\theta)}$$

   $$r(0) = r_0(\psi_0)$$

   d. This allows us to calculate $q = q(\psi_0)$

Basic Procedure for Solving the Grad–Shafranov Equation

1. We assure $r/R \ll 1$ but not zero

2. The equilibrium problem separates into two parts

   a. radial pressure balance

   b. toroidal force balance

3. Applications

   a. ohmic tokamak, circular, conducting shell

   b. ohmic tokamak, circular, vertical field

   c. high β tokamak, circular

d. flux conserving, tokamak, circular

e. noncircular tokamak

f. spherical tokamak

Ohmic Tokamak

1. This is the simplest configuration: low β , circular cross section, simple boundary conditions

2. Mathematical statement of the problem

$\Delta^*\psi = H(\psi, R)$ Grad–Shafranov equation

ψ is regular inside the plasma, (no wires, current sources in plasma)

$$\psi(b,\ \theta) = \text{const.} \leftrightarrow \text{n} \cdot \text{B} = \text{B}_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}\bigg|_{r=b} = 0$$

perfect conducting boundary condition

3. Solution: first transform from R, ϕ', Z to r, θ, ϕ . This step is exact; no expansions are used

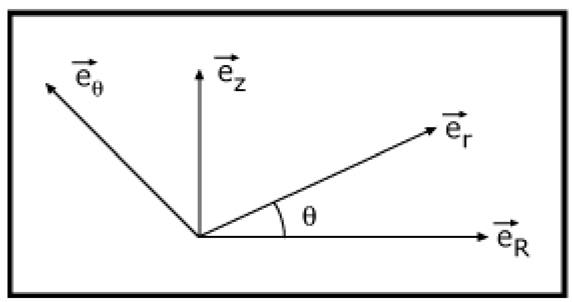
$$R = R_0 + r\cos\theta$$

$$e_R = e_r \cos\theta - e_\theta \sin\theta$$

$$Z = r\sin\theta$$

$$e_Z = e_r \sin\theta + e_\theta \cos\theta$$

$$\phi' = \phi$$



4. Then

$$\nabla\psi = \frac{\partial\psi}{\partial r}e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_\theta + \frac{1}{R}\frac{\partial\psi}{\partial\phi}e_\phi$$

$$\nabla \cdot \text{V} = \frac{1}{r}\frac{\partial rV_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{R}\frac{\partial V_\phi}{\partial\phi} + \frac{1}{R}\left(V_r\cos\theta - V_\theta\sin\theta\right)$$

$$\Delta^*\psi = \nabla \cdot \nabla\psi - \frac{2}{R}\frac{\partial\psi}{\partial R}$$

$$= \underbrace{\frac{1}{r}\frac{\partial r}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}}_{\nabla^2} - \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi_\theta}{\partial\theta}\sin\theta\right)$$

5. The Grad–Shafranov equation becomes

$$\nabla^2\psi = -\mu_0\left(R_0 + r\cos\theta\right)^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi} + \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial\psi}{\partial r}\sin\theta\right)$$

6. In the infinite aspect ratio limit

$R \to \infty$

$r/R \to 0$

$\psi(r,\theta) \to \psi(r)$ cylindrically symmetric

7. We show the Grad–Shafranov equation reduces to radial pressure balance in this limit

$$B_\phi = \frac{F}{R_0 + r\cos\theta} \to \frac{F\psi}{R_0}$$

$$B_p = \frac{1}{R_0 + r\cos\theta}\left[\frac{\partial\psi}{\partial r}e_\theta - \frac{1}{r}\frac{\partial\psi}{\partial\theta}e_r\right] \to \frac{1}{R_0}\frac{d\psi}{dr}e_\theta$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta} \to \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)$$

8. Thus the Grad–Shafranov equation becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = -\mu R_0^2\frac{dp}{d\psi} - F\frac{dF}{d\psi} \quad \text{radial pressure balance?}$$

$$T_1 = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \frac{1}{r}\frac{d}{dr}rR_0B_\theta = \frac{R_0}{r}\frac{d}{dr}rB_\theta$$

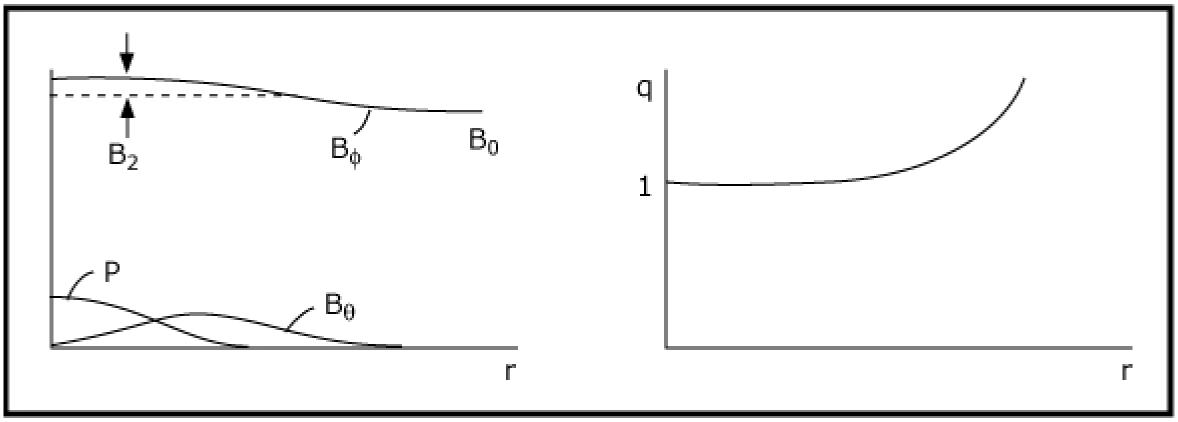
$$T_2 = -R_0^2\frac{dp}{d\psi} = -R_0^2\frac{dp}{dr}\left(\frac{d\psi}{dr}\right)^{-1} = -\frac{R_0}{B_\theta}\frac{dp}{dr}$$

$$T_3 = -F\frac{dF}{d\psi} = -\left(\frac{d\psi}{dr}\right)^{-1}\frac{d}{dr}\frac{F^2}{2} = -\frac{R_0^2}{R_0B_\theta}\frac{d}{dr}\frac{B_\phi^2}{2} = -\frac{R_0}{B_\theta}\frac{d}{dr}\frac{B_\phi^2}{2}$$

$$\frac{d}{dr}\left(p + \frac{B_\phi^2}{2\mu_0}\right) + \frac{B_\theta}{\mu_0 r}\frac{d}{dr}rB_\theta = 0$$

Ohmic Tokamak Expansion

1. Radial pressure balance



2. Expand in terms of $\in = a/R_0$. Assume $\in \ll 1$.

    a. Assume $q \sim 1$ for stability

$$q \sim \frac{rB_\phi}{RB_\theta} \rightarrow \frac{B_\theta}{B_\phi} \sim \frac{r}{R} \sim \in$$

3. Assume the plasma is confined in radial pressure balance by the *poloidal* field (due to ohmic current)

$$p \sim \frac{B_\phi^2}{\mu_0}$$

$$\beta_p \sim \frac{\mu_0 p}{B_\theta^2} \sim 1$$

$$\beta \sim \frac{\mu_0 p}{B_\phi^2 + B_\theta^2} \sim \frac{\mu_0 p}{B_\phi^2} \sim \frac{\mu_0 p}{B_\theta^2} \frac{B_\theta^2}{B_\phi^2} \sim \in^2$$

4. Assume the toroidal field does not confine much plasma. It is actually paramagnetic in practice in experiments with no auxiliary heating

$$p \sim \frac{B_\phi \delta B_\phi}{\mu_0}$$

$$\frac{\delta B_\phi}{B_\phi} \sim \frac{\mu_0 p}{B_\phi^2} \sim \in^2$$

Summary of Assumptions

1. The ohmically heated tokamak is dominated by a large toroidal field. The plasma pressure and current are small but the safety factor is of order unity. The toroidal field is required only for stability. The small toroidal current provides both radial balance and toroidal force balance.

2. The ohmically heated tokamak expansion is given by

$$\frac{B_p}{B_\phi} \sim \in$$

$$q \sim 1$$

$$\beta_t \sim \frac{2\mu_0 p}{B_\phi^2} \sim \in^2$$

$$\beta \sim \frac{2\mu_0 p}{B_\phi^2 + B_p^2} \sim \frac{2\mu_0 p}{B_\phi^2} \sim \beta_t$$

$$\beta_p \sim \frac{2\mu_0 p}{B_p^2} \sim 1$$

$$\frac{\delta B_\phi}{B_\phi} \sim \in^2$$

Solution of Grad–Shafranov Equation

1. Expand

$$\psi(r,\theta) \to \psi_0(r) + \psi_1(r,\theta) + \ldots.$$

$$\psi_1/\psi_0 \sim \in$$

$$\psi_0 \sim rR_0 B_\theta$$

2. Expand

$$F(\psi) = RB_\phi = R_0\left[B_0 + B_2(\psi)\right]$$

$$= R_0\left[B_0 + B_2(\psi_0) + \frac{dB_2}{d\psi_0}\psi_1 + \ldots.\right]$$

$$\frac{B_2}{B_0} \sim \in^2 \quad \text{(small dia/paramagnetism)}$$

3. Expand $p(\psi) = p(\psi_0) + \dfrac{dp}{d\psi_0}\psi_1 + \ldots$

4. Expand $R = R_0\left(1+\dfrac{r}{R_0}\cos\theta\right)$

5. Fields:

$$B_\theta = \frac{1}{R_0}\frac{d\psi_0}{dr} + \left[\frac{1}{R_0}\frac{\partial\psi_1}{\partial r} - \frac{r}{R_0^2}\frac{d\psi_0}{dr}\cos\theta\ldots\right]$$

$$B_r = -\frac{1}{R_0 r}\frac{\partial\psi_1}{\partial\theta}$$

$$B_\phi = B_0\left[1-\frac{r}{R_0}\cos\theta + \frac{r^2}{R_0^2}\cos^2\theta + \frac{B_2(\psi_0)}{B_0} + \ldots\right]$$

6. Expansion of the Grad–Shafranov equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = -\mu_0\left(R_0 + r\cos\theta\right)^2\frac{dp}{d\psi} - \frac{d}{d\psi}\frac{F^2}{2} + \frac{1}{R}\left(\frac{\partial\psi}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right)$$

$$\in^0\text{:}\ \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi_0}{dr}\right) = -\mu_0 R_0^2\frac{dp}{d\psi_0} - \frac{d}{d\psi_0}R_0^2 B_0 B_2$$

$$\in^1\text{:}\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi_1}{\partial\theta^2} = -2\mu_0 R_0 r\cos\theta\frac{dp}{d\psi_0} - \mu_0 R_0^2\psi_1\frac{d^2p}{d\psi_0^2} - \psi_1\frac{d^2}{d\psi_0^2}\left(R_0^2 B_0 B_2\right) + \frac{1}{R_0}\frac{d\psi_0}{dr}\cos\theta$$

$\in^0$ consequences

1. $\dfrac{d}{dr}\left(p + \dfrac{B_0 B_2}{\mu_0}\right) + \dfrac{B_0}{\mu_0 r}\dfrac{d}{dr}rB_\theta = 0$

2. $q(r) = \dfrac{rB_0}{R_0 B_\theta} + O(\in)$

$\beta_t = \dfrac{4\mu_0}{a^2 B_0^2}\displaystyle\int pr\,dr$ all are determined in leading order

$$\beta_p = \frac{16\pi^2}{\mu_0 I^2}\int pr\,dr$$

3. Equilibrium relation for a tokamak

   a. From the definitions we can find a relation between β_t and β_p

   b. $B_\theta(a) = \dfrac{\mu_0 I}{2\pi a} \rightarrow q_a = \dfrac{2\pi a^2 B_0}{\mu_0 R_0 I}$

   c. $\beta_t = \dfrac{4\mu_0}{a^2 B_0^2} \int pr\, dr = \dfrac{4\mu_0}{a^2 B_0^2} \dfrac{\mu_0 I^2}{16\pi^2} \beta_p = \left(\dfrac{\mu_0 I}{2\pi a B_0} \right)^2 \beta_p$

   d. $\beta_t = \dfrac{a^2}{R_0^2} \left(\dfrac{\mu_0 R_0 I}{2\pi a^2 B_0} \right)^2 \beta_p$

   e. $\beta_t = \dfrac{\in^2}{q_a^2} \beta_p$

4. q on axis

   a. $q_0 = \left. \dfrac{r B_0}{R_0 B_\theta} \right|_{r=0}$

   $B_\theta(r) \approx B_\theta(0) + B_\theta^{'}(0) r + B_\theta^{''}(0) r^2/2 \approx B_\theta^{'}(0) r$

   b. $q_0 = \dfrac{B_0}{R_0 B_{\theta 0}^{'}}$

   $\dfrac{1}{r}(r B_\theta)^{'} = \mu_0 J_0 \rightarrow B_\theta \approx \dfrac{\mu_0 J_0 r}{2} \rightarrow B^{'}_{\theta 0} = \dfrac{\mu_0 J_0}{2}$

   c. $q_0 = \dfrac{2 B_0}{\mu_0 R_0 J_0}$