MID-TERM EXAM SOLUTIONS

1. SHORT ANSWER

a) $\omega_{pe}^2 = \dfrac{4\pi n e^2}{m_e}$, $\omega_{pe} = \sqrt{\dfrac{4\pi n e^2}{m_e}}$

(-1 for formula squared without notation)

b) $\omega_{ci} = q_i B / m_i c$

c) $\lambda_D^2 = T_e / 4\pi n e^2$

d) FALSE : QUASINEUTRALITY FOR $\lambda \gg \lambda_D$

e) FALSE : PLASMA OSCILLATIONS EXIST ONLY FOR $\lambda \gg \lambda_D$

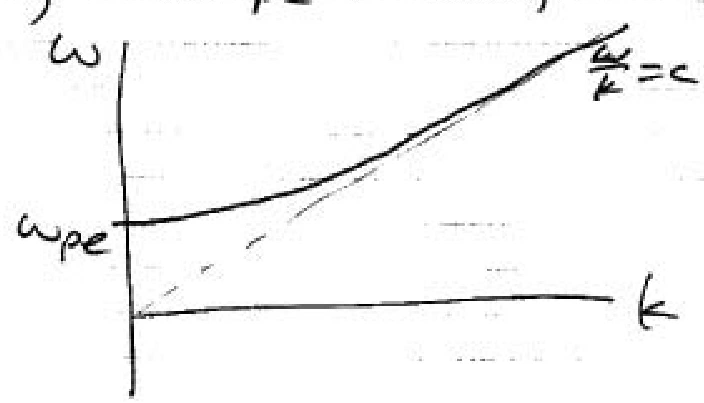
f) TRUE

g) $0 = -\nabla P + \frac{1}{c} \underline{J} \times \underline{B} = -\nabla_\perp \left(P + \frac{B^2}{8\pi}\right) + \underline{\kappa} \frac{B^2}{4\pi}$

(either or)

2. $D = 1 - \dfrac{\omega_{pe}^2}{\omega^2} - \dfrac{k^2 c^2}{\omega^2}$

a) $\omega^2 = \omega_{pe}^2 + k^2 c^2$; $\omega = \pm\sqrt{\omega_{pe}^2 + k^2 c^2}$



b) $U = \omega \frac{\partial}{\partial \omega} D \frac{|E|^2}{16\pi}$ $\quad$ $1/16\pi$ includes $\frac{1}{2}$ from ave. of sinusoid

$$\omega \frac{\partial D}{\partial \omega} = 2\left(\frac{\omega_{pe}^2}{\omega^2} + \frac{k^2c^2}{\omega^2}\right) = 2 \quad (\text{using } D=0)$$

$$U = \frac{1}{8\pi}|\hat{E}|^2$$

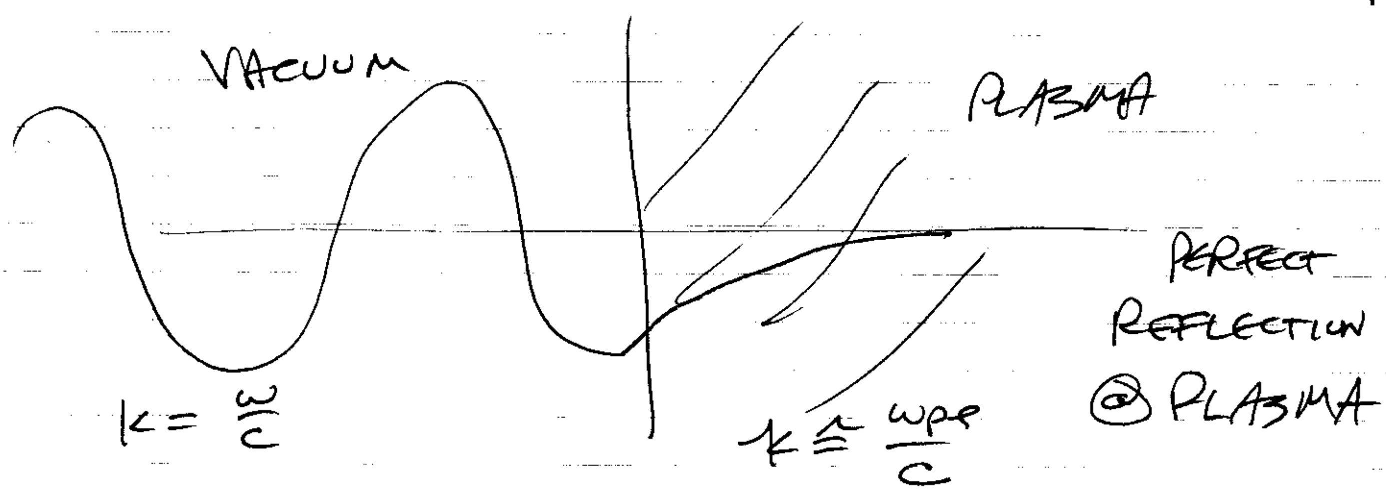
c) CUTOFF WHEN INDEX OF REFRACTION, $N \equiv \frac{kc}{\omega}$ GOES TO ZERO.

$$N^2 = 1 - \frac{\omega_{pe}^2}{\omega^2} \rightarrow 0 \text{ @ } \omega = \omega_{pe} \; (\pm)$$

is cutoff frequency.

FOR $\omega < \omega_{pe}$, $N^2 < 0$ which implies a pure imaginary, $k = \pm i K$. Such waves will EVANESCE in appropriate direction.

EXAMPLE VACUUM $\rightarrow$ PLASMA @ $\omega \ll \omega_{pe}$



PARTIAL CREDIT POINTS

3. COLLISIONS & CONDUCTIVITY

(4)

$$\sigma_{12}^{\text{COUL}} = \frac{4\pi q_1^2 q_2^2}{m_R^2 v_1^4} \ln\Lambda \qquad m_R = \frac{m_1 m_2}{m_1+m_2}$$

$$\Lambda = n\lambda_D^3 = \frac{\lambda_D}{b_{90}}$$

COLLISION FREQUENCY FROM AVERAGING OVER DISTRIBUTION:

$$\nu_{ei}^P \hat{=} n_i \langle v_1 \sigma_{ei} \rangle$$

MOMENTUM "FACTOR" $\frac{m_i}{m_e+m_i} \hat{=} 1$

$$\rightarrow n_i v_e \sigma_{ei}(v_e)$$

(4)

$$= \frac{4\pi n_i e^4 Z^2 \ln\Lambda}{m_e^2 v_e^3} \qquad v_e = \sqrt{2T_e/m_e}$$

STEADY STATE MOMENTUM EQ. BALANCES $\underline{E}$ FIELD ACCELERATION OF ELECTRONS WITH COLLISIONAL DRAG (ON IONS):

$$m_e \frac{d\underline{v}_D}{dt} \rightarrow 0 = -e\underline{E} - m_e \underline{v}_D \nu_{ei}^F$$

$\underline{v}_D$ = FLUID DRIFT VELOCITY

(4)

$$\nu_{ei}^F = \boxed{\xi}\, \nu_{ei}^P$$

is FRICTION &

$\boxed{\xi}$ is order unity numerical constant.

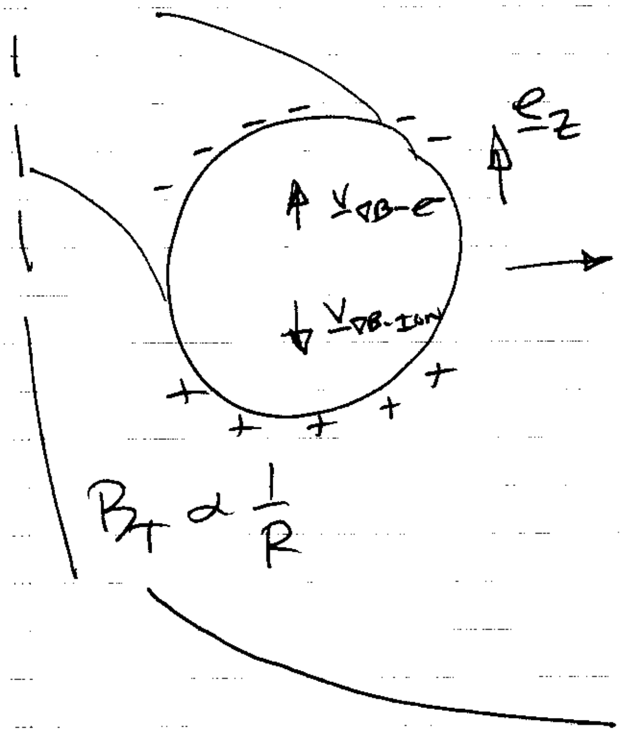
$$\underline{v}_D = -\frac{e}{m_e \nu_{ei}^F} \underline{E}$$

which gives a current:

(4) $\underline{J} = -e n_e \underline{V}_D = \dfrac{n e^2}{m_e \nu_{ei}^F} \underline{E}$

$$\boxed{\sigma = \frac{n e^2}{m_e \nu_{ei}^F}}$$ is plasma conductivity $\propto T_e^{3/2} \propto n^0$.

4. TOROIDAL DRIFTS

(5) 

CURVATURE & ∇B DRIFTS

$\underline{V}_{\nabla B} = \dfrac{c}{qB^2}(-\mu \nabla B \times \underline{B})$

$\mu = \frac{1}{2} m v_\perp^2 / B$

$\underline{V}_\kappa = \dfrac{c}{qB^2} m v_\parallel^2 \underline{\kappa} \times \underline{B}$

$\underline{\kappa} = -\underline{b} \cdot \nabla \underline{b} \doteq -\nabla_\perp B$ (for low β)

Both $\underline{\kappa}$ & $-\nabla_\perp B$ are $\parallel \underline{e}_R \Rightarrow$

~~$q \underline{V}_{\nabla B} \parallel \underline{V}_\kappa \parallel \underline{e}_R \times \underline{e}_T$~~

$q \underline{V}_{\nabla B} \parallel q \underline{V}_\kappa \parallel \underline{e}_R \times \underline{e}_T = -\underline{e}_Z \Rightarrow$

IONS DRIFT DOWN, ELECTRONS DRIFT UP

(5) (COMPARABLE SPEEDS) LEADING TO CHARGE SEPARATION AS SHOWN.

(5) RESULTING VERTICAL ELECTRIC FIELD, $\underline{E} \parallel \underline{e}_z$, CAUSES $\underline{E} \times \underline{B}$ DRIFT OUT OF ENTIRE PLASMA. SINCE CHARGE GROWS LINEAR IN TIME WITH CONTINUING $\underline{V}_{DB}$ & $\underline{V}_K$

(1) DRIFTS, $\underline{E} \propto t$, SO $\underline{E} \times \underline{B}$ DRIFT APPEARS AS ACCELERATION.

EXTRA CREDIT (+5 POINTS) The polarization drift, $\underline{V}_p = \frac{c}{B\omega_c}\frac{\partial}{\partial t}\underline{E}$, actually reduces field in plasma. If this is calculated using, $\epsilon_p = 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2}$, one can show that outward acceleration is pure CENTRIFUGAL (as though no $\underline{B}$ field present)

CALC.

$$V_K = \frac{c}{qB} m V_\parallel^2 K \Rightarrow \text{SURFACE CHARGE } \sigma_s = q n V_K t$$

$$= \frac{cn}{B} m V_\parallel^2 K t$$

$$\text{ELECTRIC FIELD: } E = \frac{4\pi\sigma_s}{\epsilon_p} \simeq \frac{\omega_{ci}^2}{\omega_{pi}^2} 4\pi\sigma_s$$

$$V_{E\times B} = \frac{\omega_{ci}^2}{\omega_{pi}^2}\frac{4\pi n c^2 m_i}{B^2} V_\parallel^2 K t = V_\parallel^2 K t$$

$$\Rightarrow \text{EFFECTIVE ACCELERATION } V_\parallel^2 K = \frac{V_\parallel^2}{R}$$

i.e. the ~~[illegible]~~ CENTRIFUGAL FORCE

5. θ - PINCH EQUILIBRIA

Easiest to take force balance in form that displays magnetic pressure and tension:

(4) $$0 = -\nabla_{\perp}\left(p + \frac{B^2}{8\pi}\right) - \underline{\kappa}\frac{B^2}{4\pi}$$

FOR $\underline{B} = B\underline{e}_z$, $\underline{\kappa} = 0$ (NO CURVATURE)

(4) So, $0 = \nabla_{\perp}\left(p + \frac{B^2}{8\pi}\right) \Rightarrow p + \frac{B^2}{8\pi} = \text{CONST.}$ = PLASMA PRESSURE + MAG. PRESSURE

APPLY BOUNDARY CONDITIONS @ $r = a$ $(p = 0)$

(4) $$p + \frac{B^2(r)}{8\pi} = \frac{B_0^2}{8\pi} \quad , \quad p(r) = \frac{B_0^2}{8\pi}\left(1 - \frac{r^2}{a^2}\right)$$

If $p(a) = p_e \neq 0$ the calculation is the same

3 $$p(r) = \frac{B_0^2}{8\pi}\left(1 - \frac{r^2}{a^2}\right) + p_e$$

This is the simplest form of equilibrium where plasma pressure is offset by MAGNETIC PRESSURE & the sum must remain constant.

6. ION BEAM - PLASMA INSTABILITY

This problem is a virtual copy of what was discussed in lecture.

LONGITUDINAL DIELECTRIC IN e-i PLASMA IS: $D = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{\omega^2}$ (WITHOUT MOTION)

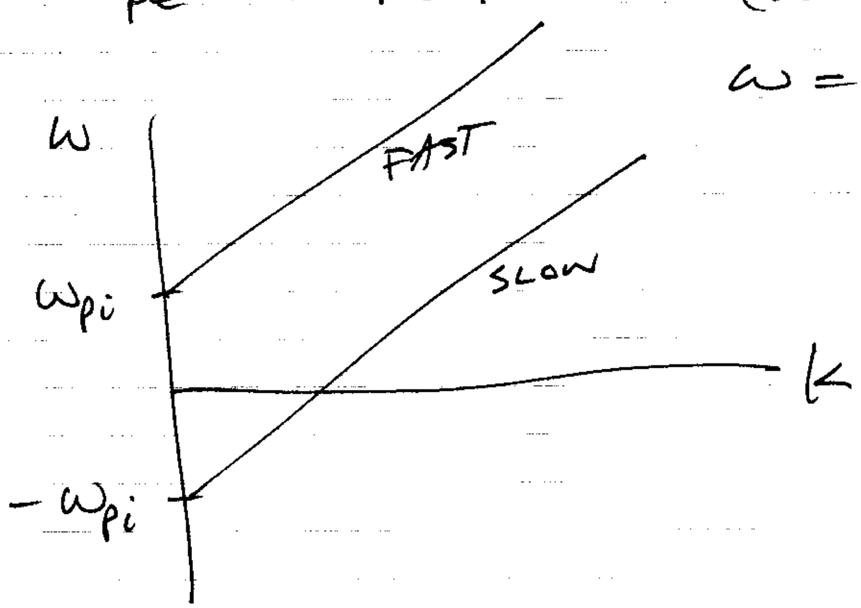
(4) If ion response is computed in its rest frame & DOPPLER SHIFTED to lab frame, the result is $\omega \to \omega - kV_0$ for the ion term:

$$D = 1 - \frac{\omega_{pe}^2}{\omega^2} - \frac{\omega_{pi}^2}{(\omega - kV_0)^2}$$

$\omega_{pe} \to 0$ RESPONSE : $(\omega - kV_0)^2 = \omega_{pi}^2 \Rightarrow$

$$\omega = kV_0 \pm \omega_{pi}$$

(4)

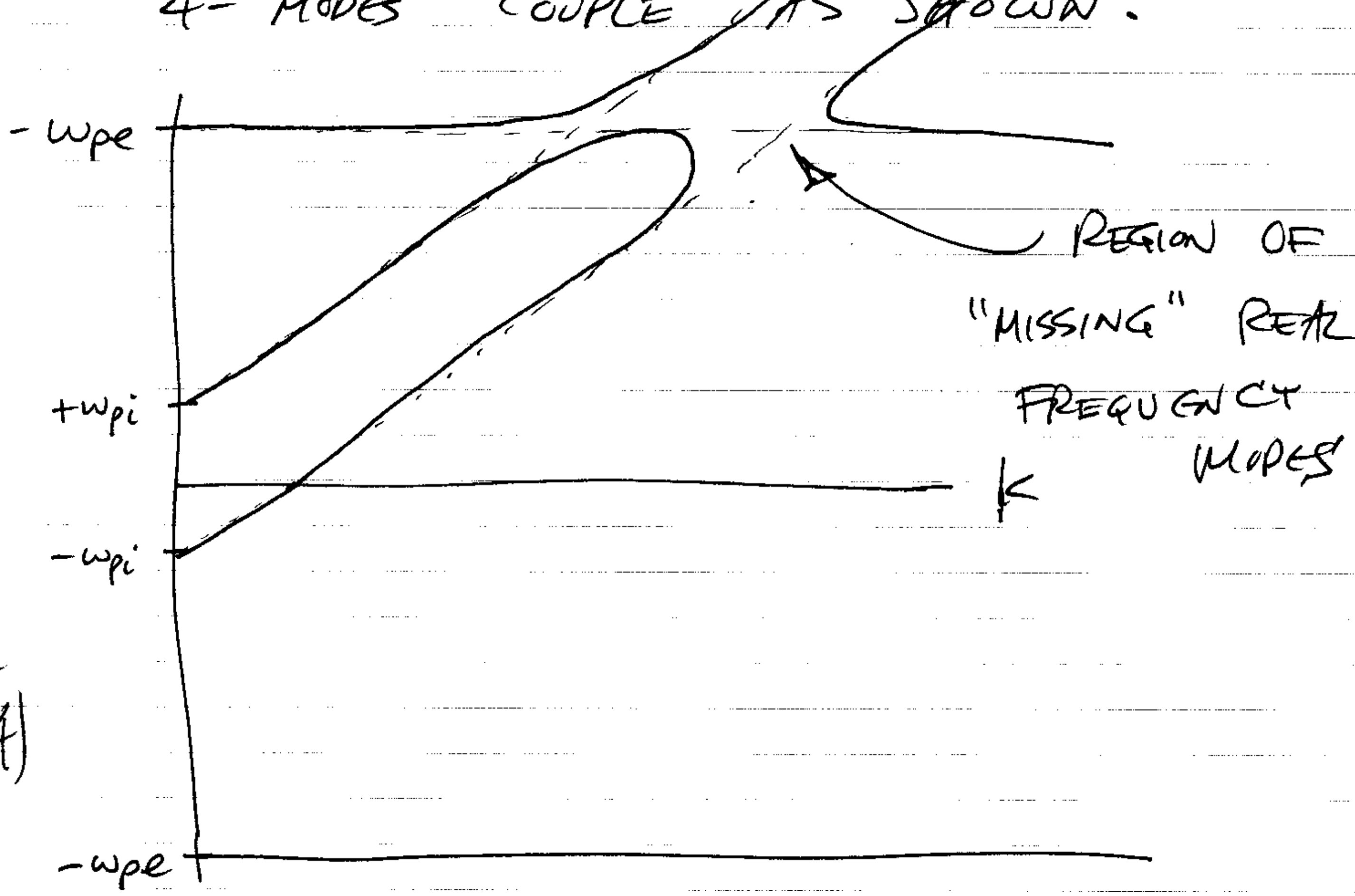


WAVE ENERGY: $\omega \frac{\partial D}{\partial \omega} \frac{|\tilde{E}|^2}{16\pi}$

(4) $$\omega \frac{\partial D}{\partial \omega} = \frac{2\omega\omega_{pi}^2}{(\omega - kV_0)^3} = \frac{2\omega}{\pm\omega_{pi}} = 2\,\frac{kV_0 \pm \omega_{pi}}{\pm\omega_{pi}}$$

This will be NEGATIVE for the SLOW WAVE (at positive frequencies)

WHEN ω_{pe} RESPONSE IS ADDED THE 4-MODES COUPLE AS SHOWN:



(4)

The disappearance of real roots indicates a COUPLING OF POSITIVE & NEGATIVE ENERGY MODES. This leads to instability since both can grow in amplitude while maintaining the ~~sys~~ system energy. (Necessary since we have NO DISSIPATION IN THIS SYSTEM)

<u>CALCULATION OF GROWTH RATE</u>

(+5 POINTS) Since $\omega_{pi}^2 \ll \omega_{pe}^2$ need $\omega \doteq kV_0$ for ion term contribution and $\omega \doteq \omega_{pe}$ to cancel 1 term: LOOK @ $kV_0 = \omega_{pe}$.

Let $\omega = \omega_{pe} + \delta\omega$

$$1 - \frac{\omega_{pe}^2}{(\omega_{pe} + \delta\omega)^2} - \frac{\omega_{pi}^2}{\delta\omega^2} = 0 \doteq 2\frac{\delta\omega}{\omega_{pe}} - \frac{\omega_{pi}^2}{\delta\omega^2}$$

$$\delta\omega^3 = \tfrac{1}{2}\omega_{pe}\omega_{pi}^2$$

$$\delta\omega = \left(\tfrac{1}{2}\omega_{pe}\omega_{pi}^2\right)^{1/3}\left(1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\right)$$

UNSTABLE ROOT