22.51 Quantum Theory of Radiation Interactions

Problem set # 4

Issued on Monday Oct 22, 2012. Due on Monday Nov. 5, 2012

Problem 1: NMR relaxation

Consider an ensemble of spin-$1/2$ particles (a liquid NMR sample) in the presence of a static inhomogeneous magnetic field $\vec{B}(z)$ that leads to the following Hamiltonian $H(z)$:

$$\vec{B}(z) = (B_0 + az)\hat{z} \implies H(z) = \frac{\hbar}{2}(\omega_0 + Gz)\sigma_z \tag{1}$$

a) Since we detect the magnetization of the whole sample as an effective one spin-$1/2$ particle, the corresponding density matrix is an average over the spatial variation:

$$\rho(t) = \int p(z)\rho(z,t)dz \ \text{ with } \ p(z) = 1/L \ \text{ for } |z| < L/2 \ \text{ (and zero otherwise.)} \tag{2}$$

where $\rho(z,t)$ is the evolution of the system under $H(z)$. Find $\rho(t)$ for an arbitrary initial state.

b) Show that this process can also be described equivalently by a Kraus set given by

$$E_1 = \sqrt{\frac{1}{2} - \frac{\sin GtL/2}{GtL}} \sigma_z \exp(-i\omega_0 t\sigma_z/2) \tag{3}$$

$$E_2 = \sqrt{\frac{1}{2} + \frac{\sin GtL/2}{GtL}} \exp(-i\omega_0 t\sigma_z/2) \tag{4}$$

What interpretation can you give to this process? Is this Kraus set unique?

Problem 2: Evolution of a TLS coupled to a two-level environment

Obtain an expression for the Kraus operators that describe the evolution of a single TLS ("system S") that is coupled to another TLS ("environment, E") which we are not interested in observing. The full evolution of the complete system is given by the Hamiltonian

$$\mathcal{H} = J\sigma_z^{(S)} \otimes \sigma_z^{(E)}$$

a) What are the Kraus operators?

b) What is the evolution of qubit S alone?

c) Now consider the hamiltonian

$$\mathcal{H}_{SE} = J\sigma_z^{(S)} \otimes \sigma_y^{(E)}$$

Find again the Kraus operators and write down the evolved state. Is the evolution of qubit S unitary in this case?

d) Compare this problem to problem 6 in PSet2. How is the character (unitary or not) of the reduced system evolution connected to the creation of entanglement between system and environment? Comment.

Problem 3: Generalized amplitude damping

Consider a map given by the following Kraus set

$$E_0 = \sqrt{p}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \sqrt{p}\begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad E_2 = \sqrt{1-p}\begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \sqrt{1-p}\begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$$

with $0 < \{\gamma, p\} < 1$

a) For an arbitrary initial state $\rho(0) = (\mathbb{1} + \vec{n}(0) \cdot \vec{\sigma})/2$, find $\vec{n}(1)$, that gives the state of the system after one step of the map $\rho(1) = S[\rho(0)]$. Visualize this in the Bloch sphere: what is happening to the Bloch vector $\vec{n}$?

b) Consider now $p = 1/2$. Find $\vec{n}(k)$ for the state after k steps. Again, visualize this in the Bloch sphere: what is happening to $\vec{n}(k)$?

c) Set $\gamma = \Gamma \delta t$ and $k = t/\delta t$. Write the system's state at a time t, $\rho(t) = S(t)[\rho(0)]$ that you found above. If the TLS has initially a polarization P (i.e. the state is $\rho = \frac{\mathbb{1} + P\hat{v}\cdot\vec{\sigma}}{2}$, with $\hat{v}$ a unit vector) what is the polarization P' of the evolved state $\rho(t)$? What is the state of the system when $t \to \infty$? What is its entropy then?

c) For unitary evolution of a state, we can invert the evolution of the system by going "back in time" with the substitution $t \to -t$. [If $U = e^{-i\mathcal{H}t}$, applying $t \to -t$ we have $e^{i\mathcal{H}t} = U^\dagger = U^{-1}$, and $\rho(-t) = U^\dagger \rho U$ is still a valid density operator].

Now consider the superoperator evolution defined above, which gives the evolved state $\rho(t) = S(t)[\rho(0)]$. What is the polarization of the evolved state if we go back in time (by inverting the sign, $t \to -t$)? Is the state still a good valid density operator?

Problem 4: Lindblad forms

Consider a two-level system governed by a Hamiltonian $\mathcal{H} = -\hbar\omega\sigma_z/2$. Its interaction with the environment is represented by a single Lindblad operator $L_1 = \sqrt{\alpha}\sigma_-$, where $\sigma_- = |0\rangle\langle 1|$ is called *lowering* operator, and α is the rate at which this process occurs.

a) Write out the Lindblad equation for this process.

b) Move to an interaction picture where the system is described by $\tilde{\rho} = e^{i\mathcal{H}t/\hbar}\rho e^{-i\mathcal{H}t/\hbar}$. Find a new Lindblad equation for $\tilde{\rho}$.

c) Proposing a solution of the form $\tilde{\rho}(t) = (\mathbb{1} + \vec{n}(t) \cdot \vec{\sigma})/2$, find $\vec{n}(t)$
[Hint: notice that in general $n_\alpha = \text{Tr}\{\rho\sigma_\alpha\}$, with $\alpha = \{x, y, z\}$].

Problem 5: Entanglement decay

Consider the paper *Sudden Death of Entanglement* by Ting Yu and J. H. Eberly,
On the first page, at the top of the 3rd column the authors state: *Entanglement of idealized cats could be denoted with a bracket such as* $[(Ws) \Longleftrightarrow (Sw)]$*, where we have chosen large and small letters to distinguish a big cat from a little cat.*

a) How would you write down this state with the usual Dirac notation? Choose a suitable basis and write down the equivalent state for two atoms being in the ground and excited states (instead of sleeping and awake).

On page 2, first column, the authors describe the decay process: *In the case of spontaneous emission there is no environment at all, except for the vacuum. The vacuum can still have a noisy degrading effect through its quantum fluctuations, which cannot be avoided, so both atoms in Fig. 1 must eventually lose their excitation and come to their ground states with the rate* Γ.

b) Consider 1 atom only. Which of the 3 processed studied in class (dephasing, polarization damping and amplitude damping) are the authors describing?
Write the Kraus operators using $p = 1 - e^{-\Gamma t}$, as in figure (2) .

c) Now assume the environment (the vacuum of the e.m. field) acts independently and simultaneously on the two

atoms. Write the Kraus operators for the composite system. [Hint: That means that you should take all the possible combinations of the Kraus for the two systems].

d) Write the system evolution in terms of these Kraus operators, for a system initially in the state $\propto |eg\rangle + |ge\rangle$.
[where e and g indicate the excited and ground state of the atoms.]

e) What is the evolution of the entanglement?
[Note: as we are dealing with a mixed state, you should calculate an entanglement measure such as the Concurrence. You can do that e.g. numerically or with Mathematica. Otherwise, for simplicity, you can just calculate the purity of the reduced system and take as an estimate of entanglement $1-$Purity$= 1 - \mathrm{Tr}\{\rho_1^2\}$, with $\rho_1 = \mathrm{Tr}_2\{\rho\}$]

Problem 6: Spin Echo with random field

We have seen in class that dephasing can arise from an inhomogeneous magnetic field, such that each element in the ensemble of spins has a slightly different frequency ω. Under the action of an external field, the spins precess and if the field is inhomogenous they acquired different phases $\varphi(t) = \gamma B(z)t = \omega(z)t$:

$$S[\rho] = \int d\varphi\, p(\varphi) e^{-i\varphi\sigma_z/2} \rho e^{i\varphi\sigma_z/2} = \left\langle e^{-i\varphi\sigma_z/2} \rho e^{i\varphi\sigma_z/2} \right\rangle$$

where $p(\varphi)$ is the probability of the phase φ.

a) Assume $\rho(0) = \frac{1}{2}(\mathbb{1} + P\sigma_x)$. What is $\rho(\tau) = S[\rho, \tau]$?

b) We now apply a π-pulse (that is, an operation that exchange $|0\rangle$ with $|1\rangle$, such as σ_x) at the time $\tau/2$, so that the total evolution is : free precession from $t = 0 \to t = \tau/2$, π-pulse, free precession from $t = \tau/2 \to t = \tau$ (the pulse takes no time). This is a spin echo sequence. What is $\rho(\tau)$ in this case?

c) How does your answer change if $\varphi(t) = \int_0^t \omega(t')dt$, with a time-varying frequency $\omega(t)$?

d) You should have found a term $\propto \left\langle \cos\left(\int_0^{\tau/2} dt\, \omega(t) - \int_{\tau/2}^{\tau} dt\, \omega(t) \right) \right\rangle$. For any Gaussian stationary variable ϑ with $\langle\vartheta\rangle = 0$, $\langle e^{i\vartheta}\rangle = e^{-\langle\vartheta^2\rangle/2}$. What is $\rho(\tau)$, if $\omega(t)$ is a Gaussian stationary variable with $\langle\omega\rangle = 0$? Could this process be described by a simple Master equation?

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