# 22.51 Quantum Theory of Radiation Interactions

## Problem set # 3

Issued on Wednesday Oct 10, 2012. Due on Monday Oct. 22, 2012

## Problem 1:  Trace of tensor products

Take a tensor product space $\mathcal{H} = \mathcal{A} \otimes \mathcal{B}$ of dimensions $m \otimes n$ and consider two operators $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Show that:

**a)** $\mathrm{Tr}\{A \otimes B\} = \mathrm{Tr}\{A\}\,\mathrm{Tr}\{B\}$

**b)** $\mathrm{Tr}\{A \otimes \mathbf{1}_B\} + \mathrm{Tr}\{\mathbf{1}_A \otimes B\} = n\,\mathrm{Tr}\{A\} + m\,\mathrm{Tr}\{B\}$

## Problem 2:  A Nobel Prize experiment

Consider the paper by Serge Haroche's group describing an experiment that follows the quantum life of a photon, Nature 446, 297-300 (2007)

Fig. 1 of the paper describes the experimental apparatus, that consists of a first Ramsey cavity $R_1$, a cavity C, a second Ramsey cavity $R_2$ and the detector D. The atoms can be in two possible states $|g\rangle$ or $|e\rangle$ (the ground and excited states).

The Box $B$ prepares a flux of atoms in their *ground state*. We will consider the evolution of a single atom from that initial state.
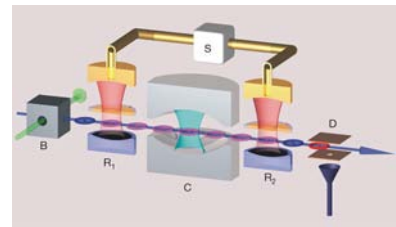


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**a)** The Ramsey cavity $R_1$ operates on the atoms with the Hamiltonian

$$\mathcal{H}_{R1} = i\Omega_R(|e\rangle\langle g| - |g\rangle\langle e|)$$

The atoms take a time $t_R = \pi/(\Omega_R 4)$ to traverse the Ramsey cavity (during which time they experience the Hamiltonian above). In the cavity C, the atom interacts for a time $t_C$ with the electro-magnetic field, with an Hamiltonian

$$\mathcal{H}_C = n\omega(|g\rangle\langle g| - |e\rangle\langle e|)$$

where $n$ is the number of photons in the cavity itself (either 0 or 1). Finally, the second Ramsey cavity $R_2$ has an Hamiltonian that is similar to $R_1$:

$$\mathcal{H}_{R2} = -i\Omega_R(|e\rangle\langle g| - |g\rangle\langle e|)$$

and it takes the same time $t_R = \pi/(\Omega_R 4)$ for the atom to traverse this Ramsey cavity.

What is the atom's state after the first Ramsey cavity?
What is the probability of finding the atom in the state $|g\rangle$ at the detector D?

**b)** Now consider the e.m. field in the cavity C as a quantum mechanical system. Under the experiment conditions, the field is such that there's at most one photon in the cavity (that is, either 0 or 1 photons). With $H_A$ the atom and $H_{e.m.}$ the photon Hilbert space, what is a basis for the Hilbert space $H = H_A \otimes H_{e.m.}$? What is the Hamiltonian $\mathcal{H}_C$ for the interaction of the atom with the photon in the cavity?

**c)** Assume that initially we do not know how many photons are in the cavity, but only that it is more probable to have zero photons (with 75% of probability). The atom is in the state you found in question (a). What is the state of the atom-photon system at the instant in which the atom enters the cavity C?

**d)** The atom stays in the cavity for a time $t = \pi/(2\omega)$. We then detect the atom (at the detector D after passing in the Ramsey cavity $R_2$) and find that it's in the state $|e\rangle$. What does this tell us about the number of photons in the cavity?
*[Hint: You might be able to give an answer even without doing any calculations.]*
*This is the method used in the paper to indirectly measure the photon number (either 0 or 1) by looking at the state of the atom. Even if we do not know a priori the number of photons in the cavity, we can infer it from the measurement of the atom's state.*

**e)** If we now assume the field is initially in a superposition state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ (and the atom still in the ground state $|g\rangle$ before the first Ramsey cavity), is the final state entangled?
What would be the correlation between measuring the atom in the excited (ground) state and measuring one (zero) photons?

## Problem 3: Tripartite entanglement

Consider the GHZ and W states of three spin$-\frac{1}{2}$. The GHZ state is defined as $|GHZ\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$ while the W state is: $|W\rangle = (|100\rangle + |010\rangle + |001\rangle)/\sqrt{3}$. These are representatives of two independent classes of entangled state for tripartite systems (that is, any tripartite entangled state can be rotated into either a GHZ or W states by *local* operations on the single qubits, but we cannot go from one GHZ state to a W state with local operations).

**a)** What is the final state of the two remaining spins if you loose (trace over) one spin ?

**b)** What is the purity and entanglement of the reduced states?

## Problem 4: Electron and Nuclear Spins: Magnetization

Consider an electron with spin-$\frac{1}{2}$, placed in a large magnetic field $\vec{B} = B_z \vec{z}$ at zero temperature. The spin is in the state
$$|\psi\rangle = c_0|\uparrow\rangle + c_1|\downarrow\rangle$$
where $|\uparrow\rangle \equiv \left|S_z = +\frac{\hbar}{2}\right\rangle$ ($|\downarrow\rangle \equiv \left|S_z = -\frac{\hbar}{2}\right\rangle$) represents the spin state aligned (anti-aligned) with the z-axis.

**a)** If the the spin is in its ground state, what are $c_0$ and $c_1$? What is the average magnetization $\langle \mu_z \rangle$?
[Assume that the only interaction is the Zeeman interaction, $\mathcal{H}_Z = \hbar\gamma B_z \sigma_z/2$]

**b)** Now assume that we cannot achieve zero temperature, but only the liquid Nitrogen temperature $T = 77$K, so that the spin is at thermal equilibrium in the field $\vec{B} = B_z \vec{z}$. What is the state of the spin?

**c)** What is the average magnetization? How many spins would you need in order to achieve the same magnetization as in question (a) if the magnetic field is $B_z \approx 6$T?

**d)** Consider now nuclear spins in a molecule. Each molecule has two spins 1/2 that interact via an Hamiltonian $\mathcal{H} = J\sigma_z^1\sigma_z^2/4$, while we can neglect the interaction among spins of different molecules. The molecules are at room temperature in a large magnetic field $B_z$. What is the equilibrium density operator and polarization in the high temperature limit?

## Problem 5: Measurement and Decoherence

We are interested in describing the quantum behavior of a particle –such as an atom or a ion– which is confined in a double well potential. The particle can be in either the left or right well. It has energy $E_R = E_0 - \Delta E$ in the right well and energy $E_L = E_0 + \Delta E$ in the left well. The barrier between the two wells is not very high, so the particle can tunnel between the two wells at a rate $\Gamma$. We neglect all degrees of freedom and characteristics of the particle, except for its position (either left or right). You can set for simplicity $E_0 = 0$ and $\hbar = 1$.



Figure 1: Left: Double well system. Right: Double well with $\Delta E = 0$ and photons going through the left well.

**a)** What is the Hamiltonian describing the tunneling, $\mathcal{H}_T$? What is the total Hamiltonian – describing the tunneling and the energy of the particle in the two wells?

**b)** To measure if the particle is in the left well, we send in a photon. The photon, initially in the ground state $|0\rangle$, interacts with the particle (only if the particle is in the left well) and is promoted to the state $|1\rangle$. (If the photon had been in the state $|1\rangle$ the scattering would have demoted it to the state $|0\rangle$). What is the unitary operator $U$ describing the combined transformation of the particle and photon? Is measuring the state of the photon after this interaction a good way of measuring the particle's position?

**c)** Assume the particle is initially in the state $|\psi\rangle = (|R\rangle + |L\rangle)/\sqrt{2}$. If the photon (initially in the ground state $|0\rangle$) interacts with the particle, is the combined state of the photon and particle entangled? (assume the photon-particle interaction happens instantaneously).

**d)** Now assume that the photon detector we used in (a-b) is faulty: it absorbs the photon but does not record the state of the photon. What is the reduced state of the particle after the measurement? (here again, assume that the particle was in the state $|\psi\rangle = (|R\rangle + |L\rangle)/\sqrt{2}$ before the interaction with the photon).

**e)** As our photon detector is faulty, we decide not to measure the particle. Unfortunately, we cannot completely block the photons from entering the left well: there is still a flux of photons with intensity $I$. Each photon, initially in the ground state $|0\rangle$ interacts with the particle (only if the particle is in the left well) and with a probability $p = I\delta t$ is scattered by the particle and is promoted to the state $|1\rangle$. As we cannot measure the state of the photons, we are interested in the evolution of the particle only. Describe the particle's evolution (due to the interaction with the photon only, neglect the particle Hamiltonian) in terms of a Kraus operator sum.

**f)** If the initial state of the particle is given by $\rho_0 = \begin{pmatrix} \rho_{LL} & \rho_{LR} \\ \rho_{RL} & \rho_{RR} \end{pmatrix}$, what is the particle's state $\rho_n$ after the interaction with $n$ photons? (Assume that photons interact one at a time).

**g)** In the limit $\delta t \to 0$, write $\rho_n$ as a function of the time $t = n\delta t$ (remember $p = I\delta t$).

**h)** Assume $\rho_0$ is given by the state $|\psi\rangle = (|R\rangle + |L\rangle)/\sqrt{2}$. What is the state of the particle for $t \to \infty$? How does it compare to what you found in d? Comment.

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