22.38 PROBABILITY AND ITS APPLICATIONS TO RELIABILITY, QUALITY CONTROL AND RISK ASSESSMENT

Fall 2004

CONVERGENCE OF BINOMIAL AND NORMAL DISTRIBUTIONS FOR LARGE NUMBERS OF TRIALS

We wish to show that the binomial distribution for m successes observed out of n trials can be approximated by the normal distribution when n and m are mapped into the form of the standard normal variable, h.

$$\underset{\substack{\uparrow \\ \text{Binomial} \\ \text{Distribution}}}{P(m,n)} \cong \underset{\substack{\uparrow \\ \text{Normal} \\ \text{Distribution}}}{\text{Prob.}\,(h)}\ \text{, where} \tag{1}$$

Binomial Distribution:

$$P(m,n) = \binom{n}{m} p^m q^{(n-m)}\ , \tag{2}$$

$$p + q = 1\ ,\ \text{where} \tag{3}$$

p = probability of success in a single trial, and
q = probability of failure in a single trial.

Normal Distribution:

$$\text{Prob.}\,(h) = \frac{e^{-\left(h^2/2\right)}}{\sqrt{2\pi}\ \sigma}\ , \tag{4}$$

$$h \equiv \left(\frac{m-\mu}{\sigma}\right)\ , \tag{5}$$

$$\mu = np\ , \quad \text{(Binomial Distribution Mean)} \tag{6}$$

$$\sigma = \sqrt{npq}\ . \quad \text{(Binomial Distribution Standard Deviation)} \tag{7}$$

Recall Sterling Approximation: $m! \cong \sqrt{2\pi m}\ m^m\ e^{-m}$

$$\Rightarrow \binom{n}{m} \equiv \frac{n!}{m!(n-m)!} \cong \frac{1}{\sqrt{2\pi}} \left(\frac{n}{m(n-m)}\right)^{1/2} \left(\frac{n}{m}\right)^m \left(\frac{n}{n-m}\right)^m\ .$$

$$P(m,n) \cong \frac{1}{\sqrt{2\pi}} \left(\frac{1}{m(n-m)}\right)^{1/2} \left(\frac{np}{m}\right)^m \left(\frac{nq}{n-m}\right)^{(n-m)}$$

$$\cong \frac{1}{\sqrt{2\pi(npq)}}\left(\frac{np}{m}\right)^{m}\left(\frac{nq}{n-m}\right)^{(n-m)}.$$

The result above uses the relationships:

$$m = np + h\sqrt{npq}$$

$$(n-m) = nq - h\sqrt{npq}$$

to obtain result

$$\left(\frac{m(n-m)}{n}\right) = n\left(p + h\sqrt{\frac{pq}{n}}\right)\left(q - h\sqrt{\frac{pq}{n}}\right)$$

$$\cong npq\ .$$

Then, use expansion of $\ell n(1+x) \cong x - \frac{x^2}{2}$, about x=0 to evaluate Eq. 1, using $\sqrt{2\pi npq}$ Prob. (h) as:

$$-\ell n\left(\sqrt{2\pi npq}\ \text{Prob.}\ (h)\right) \cong -\ell n\left(\sqrt{2\pi npq}\ P(m,n)\right)$$

$$= \ell n\left[\left(\frac{np}{m}\right)^{m}\left(\frac{nq}{n-m}\right)^{(n-m)}\right]$$

$$= \left(np + h\sqrt{npq}\right)\ell n\left(1 + h\sqrt{\frac{q}{np}}\right) + \left(nq - h\sqrt{npq}\right)\ell n\left(1 - h\sqrt{\frac{p}{nq}}\right)$$

$$\cong \left(np + h\sqrt{npq}\right)\left(h\sqrt{\frac{q}{np}} - \frac{h^2 q}{2np}\right) + \left(nq - h\sqrt{npq}\right)\left(-h\sqrt{\frac{p}{nq}} - \frac{h^2 p}{2nq}\right)$$

$$= \left(h\sqrt{npq} - q\frac{h^2}{2} + qh^2\right) + \left(-h\sqrt{npq} - p\frac{h^2}{2} + ph^2\right)$$

$$= \underbrace{(p+q)}_{1}\frac{h^2}{2} = \frac{h^2}{2}.$$

Thus, the result is obtained; verifying Eq. 4:

$$\text{Prob.}\ (h) = \frac{e^{-\left(h^2/2\right)}}{\sqrt{2\pi npq}} = \frac{e^{-\left(h^2/2\right)}}{\sigma\sqrt{2\pi}}.\ \ \text{QED!}$$