Exercise 5. Diffusion and Parabolic Equations.

1. Write a computer code to solve the diffusive equation

$$\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} + s(x)$$

For constant, uniform diffusivity D and constant specified source $s(x)$. Use a uniform x-mesh with N_x nodes. Consider boundary conditions to be $\psi = \psi_1$ at $x = 0$ and $\frac{\partial \psi}{\partial x} = 0$ at $x = 1$ (the domain boundaries).

Construct a matrix $\mathbf{D} = D_{ij}$ such that $\mathbf{D}.\psi = \nabla^2 \psi$. Use it to implement the FTCS scheme

$$\psi^{(n+1)} = (\mathbf{I} + \Delta t \mathbf{D})\psi^{(n)} + \Delta t\, s,$$

paying special attention to the boundary conditions.

Solve the time-dependent problem, for $t = 0 \to 1$, when $D = 1$, $s = 1$, $N_x = 50$, $\psi_1 = 0$, with initial condition $\psi = 0$ at $t = 0$ storing your results in a matrix $\psi(x,t) = \psi_{j_x, j_t}$, and plotting that matrix at the end of the solution, for examination.

Experiment with various Δt to establish the dependence of the accuracy and stability of your solution on Δt. In particular,

(i) find experimentally the value of Δt above which the scheme becomes unstable.

(ii) estimate experimentally the value of Δt at which $\psi(t = 1)$ is accurate to 1%.

2. Develop a modified version of your code to implement the θ-implicit scheme:

$$(\mathbf{I} - \Delta t \theta \mathbf{D})\psi^{(n+1)} = (\mathbf{I} + \Delta t(1-\theta)\mathbf{D})\psi^{(n)} + \Delta t\, s,$$

in the form

$$\psi^{(n+1)} = \mathbf{B}^{-1}\mathbf{C}\psi^{(n)} + \mathbf{B}^{-1}\Delta t\ s$$

Experiment with Δt and different θ values, for the same time-dependent problem and find experimentally the value of θ for which instability disappears for all Δt.

Also choose a Δt value for which the FTCS ($\theta = 0$) scheme is stable; then find experimentally the approximate optimum value of θ (at that fixed Δt) which produces the most accurate results.

Submit the following as your solution for each part:

a. Your code in a computer format that is capable of being executed, citing the language it is written in.

b. The requested experimental Δt and/or θ values.

c. A plot of your solution for at least one of the cases.

d. A brief description of how you determined the accuracy of the result.