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**MICHAEL SHORT:** You guys asked to do some numerical examples of the stuff that we've been learning today, and I've got a fun one for you. A very real one, because it happens all the time in reactors all over the world. So to set the stage for this, hey, suppose I had-- just suppose, I had a radioactive cobalt-60 source, that was calibrated in March, 2011 to be approximately one microcurie.

And it's now October 2016. So let's start off with the easy part. Let's say I posed the question, how active was this source actually when it was made? Because it just says-- what does it say-- 1 microcurie. Supposedly, this was 1 microcurie on March, 2011.

What does that mean the actual activity could be? There's not a lot of confidence in that number. Remember significant figures from high school chemistry and physics. This is when they're important, because you got to know what you're buying. So what is the actual activity of this source when it was calibrated? What is our uncertainty on this?

Anyone remember this from sig figs? I hope so. If not, I'll refresh your memory. It's plus or minus the next decimal point. So really, this was 1 plus or minus 0.5 microcuries. So you might have a 1/2 microcurie source. You might have a 1 and 1/2 microcurie source. They specifically decided to leave out the second decimal point so they're not liable for a source that's out of that calibration level.

So we supposedly have a 1 microcurie source. And I've measured it on-- let's say it's now-- October, 2016. Suppose I went and made a measurement, and it was 0.52 microcuries. And we want to know, how active was this source actually when we got it? Where would I begin?

**AUDIENCE:** Look up the half-life.

**MICHAEL SHORT:** OK, yeah, that's why I've got the table of nuclides right here. So let's look up the half-life of cobalt-60. And there it is right there, 1,925 days. So we know that the half-life for cobalt-60 is 1,925.4 days, which equals-- and I pre-did the math out so I wouldn't spend lots of time on the

calculator.

Activity, how many seconds is that?  $1.66 \times 10$  to the eighth seconds. And we also remember that activity equation, where the activity as a function of time equals the original activity times  $e$  to the minus  $\lambda t$ . The only thing missing here is the initial activity and  $\lambda$ , so how do we find  $\lambda$ ? Anyone remember that expression?

**AUDIENCE:** Log of 2 over half-life.

**MICHAEL SHORT:** Yep. And we know that  $\lambda$  equals log of 2 over the half-life, which in this case is about 0.693 over  $1.66 \times 10$  to the eighth, which equals  $4.17 \times 10$  to the minus 9 per second. And so now it's pretty easy. We know what  $\lambda$  is. We know what our current  $A$  is. So we can say that our original activity is simply our current activity divided by  $e$  to the minus  $\lambda t$ . What's  $t$ ?

Well, I made an approximation here. It's been about five years and seven months. I assumed that a month has 30 days, on average, which comes out to  $t$ -- I remember calculating this already--  $1.74 \times 10$  to the eighth seconds. So we plug in that  $t$  right here, plug in that  $\lambda$  right there, and we get our initial activity-- what did I get?-- was 1.07 microcuries.

Now hopefully, this result is fairly intuitive. Because the half-life of cobalt-57 is 1,925 days, which is about 5 and  $1/4$  years. And it's just over 5 and  $1/4$  years since this source was calibrated. And we have just under half of the original activity. So hopefully that's an intuitive numerical example.

Now, let's say, how many atoms of cobalt-60 did we have? Or better yet, what was the mass of cobalt-60? Where would I go for that? I'll give you a hint. It's on the screen.

**AUDIENCE:** Are you saying the mass that we have in the pellet?

**MICHAEL SHORT:** Yeah, so for this plastic check source right here, what's the actual mass of cobalt-60 that we put in in the beginning, again, supposing I had one of these right in front of you?

**AUDIENCE:** Look at the binding energy.

**MICHAEL SHORT:** You could look at binding energy, which is a form of mass. Or luckily, we've got the atomic mass right up there in AMU. So again, this is a quick review of high school chemistry. We need to find out how many atoms we have in this pellet. And once we know the number of atoms--

and we have the molar mass up there in either AMU per atom or same thing as moles per gram, pretty much-- we'll know what the mass of cobalt-60 was.

So one important conversion factor to note is that 1 curie of radiation is 3.7 times 10 to the 10 becquerel. And remember that 1 becquerel is 1 disintegration per second. So if we know the initial activity of our material, using our decay constant, we should know the number of atoms that we had right there.

So we know our initial activity,  $A_0$  is 1.07 microcurie. And anyone remember, what's the relation between the activity and the number of atoms present? Just yell it out. I hope you'd know that by now. Does this look familiar to anyone? Where the activity is directly proportional to the number of atoms there, times it's decay constant.

So since we now know  $A_0$ , we can find  $N_0$ , because now we know  $\lambda$  as well. So the number of atoms we had at the beginning is just, our activity over  $\lambda$ . And our activity is 1.07 microcuries. Let's convert up to curies. So we know that there is 1 curie in 10 to the 6 microcuries. And we'll use that conversion factor right there, times 3.7, times 10 to the 10 becquerel per 1 curie, divided by our decay constant, 4.17 times 10 to the minus 9 per second.

Let's check our units to make sure everything comes out. I brought a canceling color. We have microcuries on the top, microcuries on the bottom, curies on the top, curies on the bottom. And we have a becquerels, which is a disintegrations per second. And we have a per second down on the bottom. So the per second goes away and the becquerels just becomes atoms.

And so we actually get,  $N_0$  is-- I think it's something times 10 to 12th. Yeah, 9.5 times 10 to the 12th atoms. That's the way it looks. So the final step, how do we go from number of atoms to mass? Can anyone tell me?

**AUDIENCE:** Convert it to Avogadro's number and moles.

**MICHAEL SHORT:** Yep. So the last thing we'll do is we'll say we have 9.5 times 10 to the 12th atoms, times Avogadro's number, which is 1 mole in every 6 times 10 to the 23rd atoms. Then we go to the table of nuclides to get its atomic mass.

This is one of those situations where you don't need to take the eighth decimal place, because you're not converting from mass to energy. You're just getting mass. So if you might wonder, where did all the decimal points go, think of what type of calculation we're doing. If we started

off with a one significant digit number, do we really care about keeping the eighth decimal place in the rest of everything? No, definitely not.

We're not turning AMU into MEV, in which case the sixth decimal point could put you off by half an MEV, or like the rest mass of the electron. We're just getting masses here. So let's just call that 59.9-- that's enough for me-- 59.9 grams in 1 mole of cobalt-60. And just to confirm, we have atoms here, atoms there, moles there, moles there. We should get a mass in grams.

And this came out to 0.95 nanograms of cobalt-60. Not a lot of cobalt-60 can pack quite a wallop in terms of activity. So even though this has a pretty long half-life, as far as isotopes go, it takes very little of it to have quite a bit of activity, certainly enough for our fairly inefficient handmade Geiger counter to measure what's going on. Pretty neat, huh? Cool.

Now, let's say I ask you another question, a simpler question. How many disintegrations per second are coming out of that cobalt-60 source? Let's say we wanted to find the efficiency of our Geiger counter. You'd have to know how many counts you measure. You'd have to know how far away your source is. And you'd have to know how many disintegrations per second are happening.

So how would I get to the number of disintegrations per second?

**AUDIENCE:** Convert it to becquerels.

**MICHAEL SHORT:** That's right. We'll just take our current activity, which is 0.52 microcuries. And so we'll say, 0.52 times 10 to the minus 6 curies, times 3.7, times 10 to the 10 becquerel in 1 curie is-- well, we have curies here. We have curies there. And this comes out to-- what was our current activity. It was in the tens of thousands. About 19,000 becquerels.

So we know that, right now, this source is giving off about 19,000 disintegrations per second. Is that how many gamma rays it's giving off? Do we know that yet?

**AUDIENCE:** No.

**MICHAEL SHORT:** Why not? I heard a lot of no's. What other information do we need to know?

**AUDIENCE:** Type of radiation.

**MICHAEL SHORT:** Sure. Luckily we've got that right here. So if you look here, the mode of decay is beta decay to nickel-60. So cobalt-60 is actually primarily a beta source. However, it's used for its

characteristic gamma rays.

So let's take a quick look. We look at its decay diagram, it's pretty simple. And let's just say, somewhere near 100% of the time, it beta decays up to this energy level and can undergo any number of transitions like this. The only two we really tend to see is, like that one and that one are the most likely ones.

So on average, each disintegration of a cobalt-60 atom is going to produce two highly energetic gamma rays. So actually, what you'd have to know for this source is, despite it being 19,000 becquerels of cobalt-60, it's giving off 38,000 gamma rays per second. So that way your source calculations wouldn't be off by a factor of 2.

Then if you know the distance between your source and your detector, and you know how many of those gamma rays are going through the detector itself-- which we can count calculate with a solid angle formula, which I'll give you a little later-- you'll know how many of them should interact in here, once you learn photon nuclear interactions. And you'll know how many of them actually get captured. And that's how you can get the efficiency of the detector.

So we've kind of filled in half the puzzle. You now know, for a fixed source, how many atoms there are, how many disintegrations there are and how many gamma rays you expect it to give off. And then later in the course, we'll tell you how to figure out how many of them make it into the detector and how many of those should interact in the detector. Is everyone clear on what we talked about here? Yep.

**AUDIENCE:** Where did you get the number for 38,000 gamma rays?

**MICHAEL SHORT:** Double the number of cobalt disintegrations.

**AUDIENCE:** Because we assume it drops 2?

**MICHAEL SHORT:** Yeah. So I'm ignoring the 0.2% and 0.6% decays because they're extremely unlikely. And having looked this up ahead of time, I know that this transition and that transition are by far the most likely ones. And if you don't know that--

**AUDIENCE:** Yeah, does it say it down there?

**MICHAEL SHORT:** It sure do. Those right there, the 99.9-something intensity ones, it's all there. So I had some questions on Piazza. What happens when we can't read the pixelated decay diagram? That's

just there for fun. The actual table that will tell you all the information you need to know is right below.

And I just decided, all right, forget the ones that are  $2e$  to the minus 6 likely or point 0.007, whatever, too many 0's. So you can say that, on average, to probably two or three significant digits, that gamma ray, that gamma ray are the only ones you tend to see. Is everyone clear on how I made that determination? Cool. Yes.

**AUDIENCE:** So are disintegrations just a beta, and then for each one of those, it's two gamma rays?

**MICHAEL SHORT:** That's right. The number of disintegrations is the number of atoms that leave this position. So let's take a crazier example. I've already had you look at americium-241. Sort of bring home the message that the number of americium disintegrations does not necessarily equal the number of gamma rays that you will expect to see.

Because any one of these is possible. There is an 84% likely one. It looks like it goes to the third level from the bottom, because it's just the third number I'm seeing here. If you don't believe that, then check how much you have to zoom out to see what's going on.

Let's see. Intensity, yep. The third most energetic alpha ray, so the third from the bottom, is indeed the 84% likely one. Then there's another one that's 13% likely, so we can't discount that either. So you're going to hit anything from, like, the second to the fourth energy level and any number of those gamma cascades that come off there. So the intensity of your source in becquerels or curies does not immediately tell you how many gamma rays or other disintegration products you should expect.

There are some pretty simple ones, like the dysprosium one that we saw. Was it 151? I've been here before. It's not purple. No, that's a complicated one. All right, I don't remember which dysprosium isotope just had a single decay. But the only time that you would get one particle per disintegration is if your decay diagram looked like this.

That's your parent and that's your daughter. That's the only time you can expect one thing. If you have a more complex decay diagram than that, you'll be looking at more than one quantum of radiation of some form per disintegration. So is that unclear to anybody? Cool.

Now let's get into a more fun problem. And I'll leave this up here, since we're going to refer to it a fair bit. So this is a good example of activity, half-life, mass, number of atom calculations. We're going to use this information to answer a more interesting question. How was this

made?

Anyone have any idea what sort of intentional nuclear reaction could have produced cobalt-60? And I'll go back to cobalt-60 so we can get its proton number. There it is. Can you say it a little louder?

**AUDIENCE:** [INAUDIBLE] Oh, I said neutron bombardment of cobalt-59.

**MICHAEL SHORT:** Sure, you can have a neutron bombardment. So you can have, cobalt-59 absorbs a neutron, becomes cobalt-60 with some half-life. And then we'll decay with some decay constant  $\lambda$ , and become-- in this case, it undergoes beta decay-- to nickel-60. Let me make sure I got the proton number right. I'm going from memory here. Hooray.

OK, how do we set up the series radioactive decay and production equations to describe this phenomenon? So we have a physical picture of what's going on here. How do we construct the differential equation model? The same way we've been doing it Friday and Tuesday. So someone, kick me off. What do we start with?

**AUDIENCE:** So the production of cobalt-59 is 0. Destruction would be  $\sigma$  times flux times the amount of cobalt.

**MICHAEL SHORT:** OK, let's make a couple of designations. Let's call cobalt-59-- I'll use another color for this, just so it's clear. We'll call cobalt-59  $N_1$ . We'll call cobalt-60  $N_2$ . We'll call nickel-60  $N_3$ , just to keep our notation straight.

And then continuing our notation pile here, you mentioned that there's going to be some  $\sigma$ , some microscopic cross-section times some flux, the number of neutrons whizzing about the reactor, times the amount of cobalt-60. So let's stick our  $dN_1/dt$ , equals no production minus some destruction. Does this look eerily familiar to any of you? Forget the fact that it's a  $\sigma$  and a flux. Just treat those as constants. What does this look exactly like?

**AUDIENCE:** The minus  $\lambda D$ .

**MICHAEL SHORT:** Exactly. It looks just like  $dN_1/dt$  equals minus  $\lambda$  and 1. Yeah, that's exactly it. This, in effect, is like your artificially-induced decay. So the probability that one atom of  $N_1$  absorbs a neutron, times the number of neutrons are there, gives you some rate at which these atoms are destroyed. Just like a  $\lambda$  gives you the rate at which those atoms naturally self-destruct.

So you can think of the sigma times flux like a lambda. Mathematically, they're treated identically. The only difference is, we're imposing this neutron flux. So it's like an artificial lambda, which means solving the equations and setting them up is exactly the same.

So how about  $N_2$ ? What's the production and destruction rate of cobalt-60? First of all, someone yell it out, what's the creation rate?

**AUDIENCE:** Lambda  $N_1$ .

**MICHAEL SHORT:** Which lambda?

**AUDIENCE:** Oh, lambda 1 I mean.

**MICHAEL SHORT:** So what is lambda 1, this one?

**AUDIENCE:** Sigma, yeah.

**MICHAEL SHORT:** OK, so let's keep with these notations. I'm going to say this is like a lambda artificial, but I'm going to keep with sigma flux and 1. And just so we keep our notation straight, I want to be able to cleanly separate the natural and the artificial production and destruction. And what's the destruction rate?

**AUDIENCE:** Sigma phi  $N_2$ .

**MICHAEL SHORT:** So you said sigma phi  $N_2$ . Is that the physical picture we have up here? Not quite, because like we can see here, cobalt-60 self-destructs on its own, right? Yeah. However, that actually is a correct term to put in. The other one that we're missing would be minus lambda  $N_2$ .

So you know what? Let's escalate this a bit into reality and say, we're going to do this. The actual equation is not going to be that much harder. So thanks, Sean. Let's do this. Yeah.

**AUDIENCE:** Would those sigmas be different?

**MICHAEL SHORT:** Yes, good question. That's what I was getting to. So there is an absorption cross-section for every reaction for every isotope. And now I'd like to show you guys where to find them. On the 22.01 site, there is a link to what's called the Janis Database, which is tabulated and plottable cross-sections of all kinds.

So every single database that we know of, it's updated continuously by the OECD, so you can



trust that the data is updated. I won't say accurate, because some of these cross-sections are not very well known. So I've already started it up. In this case, we don't have neutron capture data for cobalt-60. Let's keep it in the symbols for now. And then later on, we're just going to say, it's probably 0. But let's keep it together.

So for cobalt-59, double-click on that, and you are presented with an enormous host of possible reactions. Like right here, you have  $\Sigma$ , which is the total cross-section for all interactions of neutrons with cobalt-59. And you can see that this varies, but the same sort of shape that I was haphazardly drawing. It's low.

There are some resonances. And then it continuously increases at lower energies. Is this the right cross-section to use? If it accounts for every possible interaction of a neutron with cobalt-59. No. So what other reactions, besides the one that we want, which is capture, could this account for?

**AUDIENCE:** Scattering.

**MICHAEL SHORT:** Yep.

**AUDIENCE:** Fission.

**MICHAEL SHORT:** Scattering, fission,  $n_2n$  production. Sometimes one neutron goes in and two come out, like for beryllium, like we talked about at the beginning of class. So you've got to know to choose the right cross-section. And in nuclear reaction parlance-- that's shorthand right there-- that's  $\Sigma$ . That accounts for elastic scattering, inelastic scattering to any energy level, capture, fission,  $n_2n$  reactions, sometimes proton release, sometimes exploding, whatever nuclei do.

So let's look at our other choices. I'll shrink that. We have the elastic cross-section. Compare that with a total, and you get a general idea of how much the total at elastic cross-sections actually matter. So a lot of those resonances are elastic cross-section resonances, but there are other reactions that are responsible for a lot of the other craziness going on.

So let's unselect that. Oh, I'm sorry. I meant MT 1. Let's compare MT 1 and MT 2. So MT 2, elastic. There we go. OK, that's what I was hoping for. So right now, the red one is the total cross-section, and the green one is the elastic cross-section. And you can see that, at high energies, the total cross-section is mostly the elastic cross-section.

But at low energies, especially right around here at the thermal energy of neutrons in a reactor, there's something else responsible. That's probably what we're going after. So let's keep looking through this Janis Database. Hey, there's the n2n reaction, if you guys want to see how likely this is. It's another one of those reactions that-- look at that-- it's 0 until you get to 11 MEV. So what do you guys think the q-value for n2n production of cobalt-59 is?

**AUDIENCE:** Very negative.

**MICHAEL SHORT:** How negative? It's on the graph.

**AUDIENCE:** 10.

**MICHAEL SHORT:** Yeah, negative looks like 10, or maybe 10 and 1/2, MEV.

**AUDIENCE:** 0.454.

**MICHAEL SHORT:** Oh, hey, awesome.

[LAUGHTER]

**AUDIENCE:** I think it moves it around.

**MICHAEL SHORT:** So there you go. Yeah, indeed, it's very energetically unlikely to fire in one neutrons and get two. But if you have a 10.454 MEV neutron, you can make it happen. Pretty cool, huh?

That's not the reaction we're going for. What we want-- let's see if I can find it. Proton plus neutron, neutron plus deuteron, all the inelastic energy levels. There it is, capture. The gamma reaction here is what's referred to as capture. And there we go, a nice normal-looking cross-section.

So for cobalt-59, if we go down to about 0.025 eV here, read it off, it's about 20 barns. Because you guys asked for a numerical example. So let's say that our capture cross-section for cobalt-59 is about 20 barns, which is to say 20 times 10 to the minus 24th centimeter squared. Let's also put up our capture one for cobalt-60.

So let's go back to our table, take a look at cobalt-60. I think I know what the answer is going to be, which is, we don't know. Not in this database, unfortunately. So for symbolism, let's keep it there. But we're going to say, well, we don't know. Yeah.

So let's designate these different cross-sections. Let's call it sigma-59 and sigma-60. So those

will be our two cross-sections. We'll just call this one  $\sigma_{59}$ , and call this one  $\sigma_{60}$ .

And we already know the  $\lambda$  for cobalt-60. So let's say the  $\lambda$  for 60 cobalt, from this stuff up here,  $4.17 \times 10^{-9}$  per second. Let's just refer to that as  $\lambda$  for ease of writing things down. So we've got a complete set of reactions for a  $dN_2/dt$ . What about  $dN_3$ ? What's the production rate of  $N_3$ ?

**AUDIENCE:**  $\lambda$

**MICHAEL SHORT:** Yep,  $\lambda N_2$ . Anything else? What about the destruction rate? Is what?

**AUDIENCE:** [INAUDIBLE]

**MICHAEL SHORT:** It's a stable isotope.

**AUDIENCE:** We don't know.

**MICHAEL SHORT:** But you were on to it, Sean. So what would you add, based on what you added to  $N_2$ ?

**AUDIENCE:**  $\sigma$  whatever,  $\phi N_3$ .

**MICHAEL SHORT:** Yeah, there's going to be some new  $\sigma$ -- let's call it  $\sigma_{\text{nickel-60}}$ ,  $\phi N_3$ .

**AUDIENCE:** Yeah, is the  $\sigma$  going to be 0 this time around, or is it actually going to be--

**MICHAEL SHORT:** Let's find out. Let's go to our tables. There, there's data for nickel-60. Let's look up its absorption cross-section. So we'll scroll down to our  $Z$  gamma, our capture cross-section, plot it. Take a look at around  $1e^{-8}$ . And it's like 2 barns-- not negligible. So our capture cross-section for nickel-60-- I keep overwriting myself, and then I remember it's a blackboard. 2 barns, which is 2 times  $10^{-24}$  cm squared. So we can just refer to that as  $\sigma_{\text{nickel}}$ .

For the purposes of this problem, we don't particularly care how much nickel-60 we're making. Nickel-60 is a stable isotope of nickel. Eh, forget it. So for the purposes of this problem, forget the  $N_3$  equation. We don't care how much stable nickel-60 that we're making, because it's a lot cheaper to get it out of the ground, probably something like 10 orders of magnitude cheaper. Yep.

**AUDIENCE:** Where did you pick that incident energy from, the MEV [INAUDIBLE]  $1 \times 10^8$ ?

**MICHAEL SHORT:** Yeah, so the incident energy I picked-- because one thing we had gone over is that the thermal energy of a neutron is around 0.025 eV, which is equal to 2.5 times 10 to the minus eighth MEV. So I took that value, about 2.5 times 10 to the minus eighth MEV, so around here, and just went up.

And on a log scale, it looks to be closest to 2 barns. You can always get the actual value. So if you want to zoom right in, I'm going to keep zooming into the 10 to the minus eighth region. Maybe not. You can set your bounds accordingly. Oh, 0 or negative values-- ah, OK, whatever. Let's just read off the graph for now. You can use this tool to get the actual value.

But for problems like this, I think estimating it from the graph is going to be fine. When you get into 22.05 and you're like, what's the actual flux in the reactor to within 1% or something, estimating from the graph is no longer allowable. There are tabulated values of these things.

Oh, yeah, so you can actually set the plot settings, get the tables. Oh, tabulated-- there we go. So you can read off values of the cross-sections from a table like that. But for now, since we want to make sure to get this problem done, let's just stick with the graph.

So we don't care about the N3 equation. We're also going to say, well, let's not ignore  $\sigma_{60}$  yet. So how do we solve this set of equations? Let's make a little separation here. First of all, the easy one-- what's N1 as a function of time?

**AUDIENCE:** e to the minus  $\sigma_{59}$ , phi.

**MICHAEL SHORT:** e to the minus  $\sigma_{59}$ , phi, and what else?

**AUDIENCE:** N0.

**MICHAEL SHORT:** There is an N1 0, and there's a t. Because as you're burning it out, it matters how much time you have. Doesn't this look eerily similar to an N equals N0 e to the minus  $\lambda t$ ? Again, it's like exponential artificial decay, because we're burning those things out. For a fixed amount of neutrons going in, the amount that we burn is proportional to the amount that is there.

So that's the burn rate. That's the amount that's there. This is our artificial  $\lambda$ . So that equation's easy. What we really want to know is, what is N2 as a function of time? That's the \$60 million dollar question. And I'm not exaggerating there, because cobalt-60 is expensive.

So the question I'm posing to you guys on the homework-- so for those of you who have

started problem set 4, I'm going to be swapping out the noodle scratcher problem to this problem that we're going to begin together in class. And you guys are going to finish on the homework. So I'm kind of giving you help on the homework. What is  $N_2$ , the amount of cobalt-60 in your reactor, as a function of time? And what is your profit for running the reactor as a function of time?

Assuming a few things-- so let's set up some parameters. I'm going to say that the neutron flux is the same as that in the MIT reactor, which is about  $10^{14}$  neutrons per centimeter squared per second. We already have all of our lambdas. We have all of our sigmas.

I'm going to say that the cost of running the reactor is-- and I have a quote on this-- \$1,000 per day, which is the same as \$0.01 per second. Not a bad rate to stick something in the reactor, right? It's not bad at all. If you had to build your own reactor, your daily operating cost would actually be \$1 million a day for a commercial power plant. So every time a plant goes down, you lose \$1 million a day, plus the lost electricity or whatever that you have to buy.

So let's put that in there. And from the cost of this hypothetical cobalt-60 source, we know that cobalt-60 runs about \$100 per microcurie, because these sources run for about \$100. And so the eventual problem that we're going to set up here and you guys are going to solve on the homework-- and we can keep going a little bit on Friday if you want-- is, at what point, at what time do you shut off your reactor and extract your cobalt-60 to maximize your profit?

And this is an actual value judgment that folks that make cobalt-60 have to make. How long do you keep your nickel target in there and not hit diminishing returns? Because you're always going to be, let's say, increasing the amount of cobalt-60 that you make if your source is basically undefeatable, until you reach some certain half-life criterion. But it might not make financial sense to do so.

So let's start getting the solution to  $N_2$ . Let's see. So I'm going to rewrite our  $N_2$  equation and we can start solving it. I'm sorry, that's  $\frac{dN_2}{dt} = \sigma_{59} \phi N_1 - \lambda N_2 - \sigma_{60} \phi N_2$ . So how do we go about solving this differential equation?

**AUDIENCE:** Integrating factor.

**MICHAEL SHORT:** Yep, the old integrating factor. First, we want to get rid of the  $N_1$ , because that's another variable. And we've already decided right here that  $N_1$  is  $N_1(0) e^{-\sigma_{59} \phi t}$ . And we haven't specified, what's  $N_1(0)$ ? Let's do that now. The last number we'll put in is,

we started with a 100 gram source of cobalt-59.

When we write it in isotope parlance, it sounds exotic. But that's actually the only stable isotope of cobalt. So that's just a lump of cobalt from the ground that we stick in. So we know what  $N_1(0)$  is. So we can now rewrite this equation as-- let's just go with  $N_2$  prime for shorthand. And we'll put everything on one side of the equation. So we'll have, plus  $\lambda$  plus  $\sigma_{60}$   $N_2$ , minus  $\sigma_{59}$   $\phi$  times  $e$  to the minus  $\sigma_{59}$   $\phi$   $t$ , equals 0. So what is our integrating factor here?

**AUDIENCE:**  $e$  to the  $\lambda$  plus  $\sigma_{60}$   $\phi$   $t$ .

**MICHAEL SHORT:** Yeah. So it is,  $e$  to the integral of whatever is in front of our  $N_2$ ,  $\lambda$  plus  $\sigma_{60}$   $\phi$   $dt$ , which equals  $e$  to the  $\lambda$  plus  $\sigma_{60}$   $\phi$   $t$ . So we now multiply every term in this equation by our  $\mu$ , our integrating factor.

So let's say we have  $N_2$  prime times-- I'm just going to use  $\mu$  for shorthand, since it's going to take a long time to write. Plus-- let's see, that right there is like  $\mu$  prime, because if we take the integral of  $\mu$  times  $N_2$ -- let's see. Oh, yeah, this is right.

So we have  $\lambda$  plus  $\sigma_{60}$   $N_2$  times  $\mu$ , minus  $\mu$  times  $\sigma_{59}$   $\phi$   $e$  to the minus  $\sigma_{59}$   $\phi$   $t$ , equals 0. This stuff right here is like the an expanded product rule, so we can write it more simply. So we can say,  $N_2$  times  $\mu$  prime-- let's see-- equals  $\mu$   $\sigma_{59}$   $\phi$   $e$  to the minus  $\sigma_{59}$   $\phi$  times  $t$ .

So next we integrate both sides. And we get  $N_2$  times  $\mu$  equals-- let's see. Let's expand everything out now. So that stuff would be  $\sigma_{59}$   $\phi$   $e$  to the  $\lambda$  plus  $\sigma_{60}$   $\phi$ , minus  $\sigma_{59}$   $\phi$  times  $t$ . So if we integrate all of that, we're going to get  $\sigma_{59}$   $\phi$ .

I think there's an  $N_1(0)$  missing here, isn't there? Let's see. There should be an  $N_1(0)$  missing here. Yep. There's an  $N_1(0)$  that I dropped for some reason. Let's stick that back in--  $N_1(0)$ , and  $N_1(0)$  over that stuff,  $\lambda$  plus  $\sigma_{60}$   $\phi$  minus  $\sigma_{59}$   $\phi$ , times whatever is left,  $e$  to the  $\lambda$  plus  $\sigma_{60}$   $\phi$ , minus  $\sigma_{59}$   $\phi$   $t$ , plus  $C$ .

So now we can say, what's our integration constant  $C$ ? The last thing we haven't specified is our initial condition. So let's assume that when we started our reactor, there was no cobalt-60. That makes for the simplest initial condition. So we can substitute that in here.

So at  $t$  equals 0,  $N$  equals 0. So we've got  $0$  equals-- if  $t$  is 0, then that just becomes

$\sigma_{59} \phi N_{10}$  over  $\lambda$  plus  $\sigma_{60} \phi$ , minus  $\sigma_{59} \phi$ , plus  $C$ . And so that makes things pretty easy, because we know  $C$  equals minus  $\sigma_{59} \phi N_{10}$ , over that stuff that I keep saying over and over again. And then we've pretty much solved the equation.

The last thing we have to do is divide by  $\mu$ , and we'll end up with the same solution that we got on Friday and the same solution that we got on Tuesday. So I realized, the second after I said it last time, that, oh, we can't just absorb some  $e$  to the something  $t$  into our integration constant  $C$  because there's a variable  $t$  in it. So I would say, look at this derivation to know the whole solution.

And so finally we end up with-- anyone mind if I erase a little bit of this stuff up top? OK, I'll erase the decay diagram because we're not using that anymore. And hopefully everybody knows our conversion factor. So the end solution,  $N_2$  of  $t$ , would look like,  $\sigma_{59} \phi N_{10}$ , over  $\lambda$  plus  $\sigma_{60} \phi$ , minus  $\sigma_{59} \phi$ , times  $e$  to the minus-- what's  $\lambda_2$  in this case?  $\lambda$  plus  $\sigma_{60} \phi t$ , minus  $e$  to the minus  $\sigma_{59} \phi t$ .

And that right there is our full equation for  $N_2$ . Now you guys said, let's make this numerical. OK, we have every numerical value already chosen. I've already plugged these into the Desmos thing, so you can see generally how this goes. So we've solved it theoretically, so now let's make this numerical and make some sort of a value judgment, right?

We know  $\sigma_{59}$ , because we just looked that up. We know  $\phi$ . We impose that as  $10$  to the 14th neutrons per second. There it is. We know our  $\lambda$ . We don't know our  $\sigma_{60}$ , so we're just going to forget that for now.

But the point is, for everything except time and  $N_2$ , we have numerical constants for this. So once we plug it all in, I modified the Desmos example from last time to have the actual unit. So you can see that our fake  $\lambda_1$ , our  $\lambda$ -- we'll call it  $\lambda_1$ -- equals  $\sigma_{59}$  times  $\phi$ , which is 20 barns. 20 times  $10$  to the minus 24th centimeters squared, times  $10$  to the 14th neutrons per centimeter squared per second. And the centimeter squareds cancel. That becomes, to the minus 10. And we get that our  $\lambda_1$  is like 2 times  $10$  to the minus 9 per second.

Our  $\lambda_2$ , well, we already have that, 4.17 times  $10$  to the minus 9 per second. So this is one of those cases where  $\lambda_1$  approximately equals  $\lambda_2$ . So just like you see in the book, when you plug in all the numbers, you get a very similar equation. Which is to say, there's going to be some maximum of cobalt-60 produced.

And in this case, the x-axis I have in seconds, because that's the units we're using for everything. The y-axis is number of atoms. So right there, 6 times 10 to the 23rd, that's one mole. So at most, you can make up about 1/3 of a mole of cobalt-60 out of 100 moles of cobalt-59, which means you're never actually going to have one mole of cobalt-60. Because of the way that our natural and artificial decay constants work out, because they're fairly equal, you're never going to be able to convert and harvest it all. That's the numerical output of this.

Now I have another question for you. Is the top of this curve necessarily the profit point for this reactor? No, good answer. Why do you say no?

**AUDIENCE:** Well, you have to write another equation for the costs and the profits to maximize both of them.

**MICHAEL SHORT:** Exactly. That's what you guys are going to do on the homework. I think we've done the hard part together here. And so now I want you guys to decide, given those profit parameters-- and I will write them down on the Pset-- how long do you run your reactor to maximize your cobalt-60 profit?

So this is one of those examples where we did the whole theoretical derivation, we decided, yes, let's escalate the situation for reality. Everything works out just fine. The final answer, well, you just tack on this extra artificial bit of decay from the cobalt-60 being in the reactor. But the form of the equation is exactly the same. There's just a couple other constants in it for reality.

Then if you plug in all the numbers for the constants, so you pretend like that stuff is  $\lambda_2$  and that stuff is  $\lambda_1$ , there's  $\lambda_1$  again, there's  $\lambda_2$ , there's  $\lambda_1$ , and it's exactly the same equational form as the original solution that we had. When you plug in all the numbers, you get something remarkably similar to what's in the book, just scaled for actual units of atoms and seconds. Is there a question? Yeah.

**AUDIENCE:** On the homework, do you want us to just assume that  $\sigma_{60}$  is 0?

**MICHAEL SHORT:** Sure.

**AUDIENCE:** And have that all cancel out?

**MICHAEL SHORT:** If you can find it, that's great. But I couldn't find it that easily. You can hunt through the different databases in Janis to try to find it. But I'm not going to penalize you if you can't find it,



if I couldn't find it. So yeah, a lot of the homework is going to be redoing this derivation for yourself. Because I want to make sure that you can go from a set of equations that models an actual physical solution. And I guarantee you you'll have another physical solution on the exam.

Solve them using your knowledge of 1803, get some sort of a solution, which looks crazy, but it comes from straightforward math. Then plug in some realistic numbers and answer an actual question. How long should you run your reactor to maximize your profit? So it's kind of neat. We've been here one month together, and you can already start answering these value judgment questions about running a reactor.

And so again, I don't know who did, but I'm glad you asked, is this field mostly simulation? And the answer is no. You actually have to use math to make value judgments if you want to go and make isotopes. And then you go and make the isotopes. And you sell them to people like me so I can bring them into class and scare unwitting members of the public, theoretically. Yeah, hypothetical source indeed. So any questions on what we did here, from start to finish? Yep.

**AUDIENCE:** On the equation right in the middle there, where it says-- in parenthesis, it has lambda plus sigma-60--

**MICHAEL SHORT:** This one?

**AUDIENCE:** Yeah. Was the phi drop just like a--

**MICHAEL SHORT:** Oh, yeah, that was a mistake.

**AUDIENCE:** OK. I can just pull it back later.

**MICHAEL SHORT:** Yeah, it probably means I was talking and thinking at the same time and forgot to write that. But indeed, it's back everywhere else. Thank you. Yep.

**AUDIENCE:** So the lambda in the final N2 function equation, is that lambda 1, the theoretical lambda 1, fake lambda 1, or is that lambda 2?

**MICHAEL SHORT:** This lambda right here is the actual lambda for cobalt-60. Yep. This right here is just an analogy I'm drawing to say that, it's almost like that stuff is lambda 1. That's our original artificial decay constant. And this stuff here is like our lambda 2, because there's natural decay

and then there's reactor-induced destruction. Yep.

**AUDIENCE:** What is that factor of dividing by lambda [INAUDIBLE]. Where does that come from?

**MICHAEL SHORT:** That comes from this solution right here. There's another interesting bit, too. Did we necessarily say-- let's see. Did we necessarily say that-- no, never mind, that's fine. That comes from-- let's trace it through. So mu contains-- yeah, it comes from C. That's right.

So we had it over here, because that's part of our solution for-- let's see. Where would we trace it back to? It starts off here in the differential equation. It starts off here as well. So that's part of what's inside mu. OK, that's where it came from. So a mu contains this stuff. Yep.

**AUDIENCE:** And then once you integrate, it just becomes e to the that. It doesn't actually become lambda plus 65.

**MICHAEL SHORT:** It becomes e to the lambda plus 60 phi, times t.

**AUDIENCE:** Yeah, but when we do the integration of that, we don't get any factors of lambda plus 60 phi coming down.

**MICHAEL SHORT:** We do, actually. There is a mu stuck in right here. And so I just wanted to say that, expanding this term comes out to lambda plus sigma 60 phi, minus lambda sigma-59 phi, times t. So when you integrate this whole term-- and again, there's an N0 that should be there.

You do bring this whole pile in front of the t down on the bottom of the equation. So that's where it comes from. Yeah, cool. I don't think there's any more missing terms. OK, maybe time for a last question, because it is 10 o'clock.

**AUDIENCE:** This doesn't really have to do with your derivation or anything. So I'm pretty sure you also already explained this. But why can you put a cross-section, like that's a measure of probably in units, of centimeters squared. How does that [INAUDIBLE]?

**MICHAEL SHORT:** Ah, so the cross-section is almost like, if you fire a neutron at an atom, the bigger the atom appears to the neutron, the more likely it's going to hit it. So it's kind of a theoretical construct, to say, if something has an enormous cross-section, it's like shooting a bullet at a gigantic target, with a high probability of impact or interaction. Something with a small cross-section, there's still only one atom in the way, but it's like you're shooting a bullet at a tiny target and have less of a chance of hitting it. Does that makes sense? Cool.

**AUDIENCE:** So is it just a theoretical construct, or can you actually relate it to an actual physical cross-sectional area?

**MICHAEL SHORT:** You can't relate it to a physical cross-sectional area, as I know. It's not like a certain nucleus has a larger cross-sectional area. Otherwise, things like gadolinium, which has a cross-section of 100,000 barns, would just be a larger atom. And it's not. And yeah, Sean.

**AUDIENCE:** Are they determined only experimentally, or do we know some of way to calculate it?

**MICHAEL SHORT:** Good question. They can be theoretically calculated in some cases. In the Yip book, *Nuclear Radiation Interactions*, he does go over how to calculate those from quantum stuff. And so you'll get a little bit of that in 22.02, in terms of predicting the cross-section for hydrogen and the cross-section for water. And molecular water is not just the sum of its parts. That's the kind of crazy part.

Cross-sections do change when you put atoms and molecules together, just tend to be at lower energies, around thermal energies and such. Let's say, all of them probably can be theoretically calculated, just not that easily. But the really simple ones you can predict theoretically.

Predicting the resonances in those cross-sections, that's tough. Let's look at a simple cross-section, like hydrogen.

**AUDIENCE:** Can't you calculate it using simulations?

**MICHAEL SHORT:** Yes. Like if you know, let's say, the full wave function for a given atom or for all the electrons in an atom, you should be able to. So let's do N total for hydrogen. Much simpler, this is the kind of thing that can be predicted from theory quite easily. In fact, you will be doing this in 22.02.

The other one, no, I don't expect you to be able to predict this. But you will learn why the resonances are there and why they take the shape that they do. So last thing-- we did go like three minutes late, but everyone's still here. You can go if you have to, by the way. I can't keep you here.

If you want to know, if you want to make this equation more realistic and account for every possible energy in the reactor, you can make these cross-sections a function of energy, and integrate over the full energy range. And this is actually how it's done. And you will do this in 22.05, where you'll be able to take the energy-dependent cross-section in tabulated or

theoretical form, and then integrate this whole equation, and also account for the fact that the flux has an energy component.

Usually, it looks something like-- in a light-water reactor, if this is energy and this is flux, there'll be a bit of a thermal spike. There won't be much going on in the middle. I'm sorry, a fast spike, and there will be a thermal spike. And knowing how many neutrons are at every energy level, what's the probability of every neutron at energy level interacting, and what are the cross-sections at every energy level integrated over the full energy range is what gets you the accurate correct solution.

What we've done here is called the one-group approximation, where we've assumed that all the neutrons have the same energy, thermal energy, which is an OK assumption for thermal light-water reactors. And it'll get you a good estimate. The more neutrons you have at different energies, the less good that estimate becomes. Yeah.

**AUDIENCE:** Wait, so that thermal energy you gave us, like 0.02 [INAUDIBLE], that was estimated for the energy of the neutrons being fired.

**MICHAEL SHORT:** Let's say you have a neutron at about 298 Kelvin. From the Maxwell-Boltzmann temperature distribution, you can turn that temperature into an average kinetic energy. And that average kinetic energy will give you a velocity. And that velocity is around 2,200 meters per second. And that average kinetic energy happens to be about 0.025 eV.

So thermalized neutrons, like the ones flying about in the reactor, are moving quite slowly at just 2,200 meters a second, compared to the fast neutrons, which can be moving closer to the speed of light, not that close, but much, much, much closer. Cool. I'll take it as a good sign that you all voluntarily stayed a little late. So did you guys find this example useful?