22.01 Fall 2016, Quiz 2 Solutions

November 24, 2016

Quiz Instructions: Answers can be given symbolically or graphically, no calculation is necessary. ***No calculators, devices, or anything else allowed, except one double-sided, 8.5 x 11 inch sheet of paper.*** Define any intermediate variables which you need to complete the problems. Generous partial credit will be given for showing correct methodology, even if the solution is not given.

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1 (48 points) Short Answers, 6 points each

Each of these problems can be solved with one sentence, one equation, and/or one graph/picture.

1.1 Why does ionization stopping power *decrease* at energies below $E = 500\bar{I}$?

Answer: Charge neutralization

Explanation: The lower the incident energy of an incoming charged particle, the slower it moves. When it moves so slowly that it can actually collide with the electrons with which it interacts, it can pick them up and get neutralized.

1.2 What is the ratio of the ionization stopping power of an electron to that of a positron, and why? Neglect the Barkas effect, if you know what that is.

Answer: 1, because of z^2 dependence

Explanation: The formula for ionization stopping power goes something like this:

$$-\frac{dT}{dx} = \frac{something\, something\, something\, z^2 Z}{m_e c^2} ln\left(something\right) \tag{1}$$

Because the stopping power is proportional to z^2, where z is the incoming ion charge, positive and negative ions have the same stopping power! It just changes whether each infinitesimally small charged particle deflection pushes the ion/electron/positron towards or away from the interacting electron. The Barkas effect, if you're interested, has to do with differing amounts of polarization of the electrons in the traveling medium when regular matter or antimatter travels through it.

1.3 Why can you *not* just use the integral of ionization stopping power to accurately determine the range of protons in soft tissue?

Answer: More mechanisms! Elastic scattering! Bremsstrahlung radiation!

Explanation: Ionization stopping power is not the only mechanism by which charged particles slow down. They can also undergo radiation of photons (bremsstrahlung, or radiative stopping power) and elastic collisions with nuclei (nuclear stopping power). The full expression for stopping power is:

$$\frac{dT}{dx}_{total} = \frac{dT}{dx}_{nucl.} + \frac{dT}{dx}_{ioniz.} + \frac{dT}{dx}_{rad.} \tag{2}$$

1.4 Find three things wrong with this neutron transport equation, and explain physically (not just mathematically) why they are wrong. Integrating variables (*dStuff*) have been omitted for ease of reading:

$$\frac{1}{v}\frac{d\phi\left(\mathbf{r},E,\Omega,t\right)}{dt} = \frac{\nu\chi\left(E\right)}{2\pi}\int\limits_{V}\int\limits_{E}\int\limits_{\Omega'}\Sigma_f\left(E'\right)\phi\left(\mathbf{r},E',\mathbf{\Omega},t\right) + S_0\left(\mathbf{r},E,\mathbf{\Omega},t\right) \tag{3}$$

$$+\int\limits_{V}\int\limits_{E}\int\limits_{\Omega}\Sigma_s\left(E\right)\phi\left(\mathbf{r},E,\Omega,t\right)F\left(E\to E',\Omega\to\Omega'\right) - \int\limits_{V}\Sigma_a\left(E'\right)\phi\left(\mathbf{r},E',\mathbf{\Omega},t\right) - \int\limits_{S}\hat{\Omega}\cdot\nabla\phi\left(\mathbf{r},E,\mathbf{\Omega},t\right)$$

Extra credit (2 points): Find the fourth one.
More extra credit (3 points): Find the fifth one!
Answers with explanations:

1. The 2π below the neutron multiplication factor should be 4π, accounting for 4π total steradians in a unit sphere.
2. $F\left(E\to E',\Omega\to\Omega'\right)$ should be $F\left(E'\to E,\Omega'\to\Omega\right)$, as the prime (E') signifies neutrons scattering *from a different energy group*. The scattering kernel here is a *source* term of neutrons in our differential energy group.
3. Σ_a should be Σ_t, signifying that *any* interaction at all represents a loss of neutrons from our energy group. Just including Σ_a wouldn't count the scattering losses or other crazy reactions that take neutrons out of our energy group.
4. $\Sigma_t\left(E'\right)$ should be $\Sigma_t\left(E\right)$, signifying that it's neutrons inside our energy group that get lost by any interaction.
5. The surface integral should be a volume integral, because the $\nabla\phi$ term signifies that the divergence theorem has already been applied.

1.5 Use the Q-Equation to explain why hydrogen moderator is best moderator.

Answer: $\alpha = 0$
Explanation: The energy transfer is maximized during backscattering, when $\theta = \pi$:

$$\cancelto{0}{Q} = T_3\left(1+\frac{\cancelto{1}{M_3}}{\cancelto{A}{M_4}}\right) - T_1\left(1-\frac{\cancelto{1}{M_1}}{\cancelto{A}{M_4}}\right) - \frac{2}{\cancelto{A}{M_4}}\sqrt{T_1T_3\cancelto{1}{M_1}\cancelto{1}{M_3}}\,\cancelto{-1}{cos\left(\theta\right)} \tag{4}$$

$$0 = T_3\ \ 1+\frac{1}{A}\ \ +\sqrt{T_3}\ \ \frac{2}{A}\sqrt{T_1}\ \ -T_1\ \ 1-\frac{1}{A} \tag{5}$$

This is a quadratic equation in $\sqrt{T_3}$, yielding the relation $T_3 = \alpha T_1$, where $\alpha = \left(\frac{A-1}{A+1}\right)^2$. For the case of hydrogen, $A = 1$, and therefore $\alpha = 0$. That means that hydrogen can slow down neutrons the fastest, in only one collision.

1.6 What, physically, is a Compton wavelength shift, and why, mathematically, does the Compton energy shift decrease as photon energy decreases?

Answer: It's an increase in photon wavelength after backscattering, and mathematically the energy shift shrinks as the photon energy decreases (lower energy equals larger wavelength).
Explanation:

$$\Delta\lambda = \frac{h}{m_ec}\left(1-cos\left(\theta\right)\right);\qquad \theta=\pi;\qquad \Delta\lambda_{Compton\,Shift} = \frac{2h}{m_ec};\qquad \Delta E = \frac{hc}{\lambda}-\frac{hc}{\lambda'} \tag{6}$$

When $\theta = \pi$, the *wavelength shift* is fixed at a constant, but the *energy shift* depends on the original wavelength. A lower energy incoming photon has a larger wavelength to begin with, so the fixed Compton shift increases the original wavelength proportionally less, and therefore decreases the energy proportionally less as well.

1.7 What *is* the Klein-Nishina cross section physically speaking, and why does it reach a minimum at $\theta = \frac{\pi}{2}$ at very low incoming photon energies?

Answer: It's a differential angular cross section, and it reaches a minimum at $\theta = \frac{\pi}{2}$ because of the $sin^2(\theta)$ term only at very low energies.

Explanation: The Klein-Nishina cross section is as follows:

$$\frac{d\sigma}{d\theta} = \frac{k_0^2 e^4}{2m_e^2 c^4} \left(\frac{\nu'}{\nu}\right)^2 \left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - sin^2\theta\right) \tag{7}$$

For very low energies, the $\frac{\nu}{\nu'}$ and $\frac{\nu'}{\nu}$ terms basically vanish, leaving only the following approximate form:

$$\frac{d\sigma}{d\theta} = \frac{k_0^2 e^4}{2m_e^2 c^4} \left(2 - sin^2\theta\right) \tag{8}$$

This function has a clear minimum when $sin^2(\theta) = 1$, and that happens when $\theta = \frac{\pi}{2}$.

1.8 Why is an attenuation coefficient *the same thing* as a macroscopic cross section? Explain both physically and mathematically.

Answer: Same units, same mechanism.

Explanation: Both an *attenuation coefficient* (μ) and a *macroscopic* cross section (Σ) represent the total probability of any type of interaction of a beam of tightly collimated photons (μ) or any particle at all (Σ). They also have the same units, since:

$$\underbrace{\left(\frac{\mu}{\rho}\right)}_{\left[\frac{\cancel{cm^2}}{\cancel{g}}\right]} \underbrace{\rho}_{\left[\frac{\cancel{g}}{cm^{\cancel{3}}}\right]} = \underbrace{\mu}_{\left[\frac{1}{cm}\right]} \qquad \underbrace{N}_{\left[\frac{\cancel{atoms}}{cm^{\cancel{3}}}\right]} \underbrace{\sigma}_{\left[\frac{\cancel{cm^2}}{\cancel{atom}}\right]} = \underbrace{\Sigma}_{\left[\frac{1}{cm}\right]} \tag{9}$$

2 (26 points) Photon Spectral Identification

Reminder: Answers can be given symbolically or graphically, no calculation is necessary.

Answer the following questions about the two observed photon spectra from ^{60}Co, which emits two gamma rays at 1,173 keV and 1,332 keV. The spectra were obtained using a standard, small HPGe detector, and a new fiber-optic based detector (FORS), which uses plastic/glass as the active material. [1]

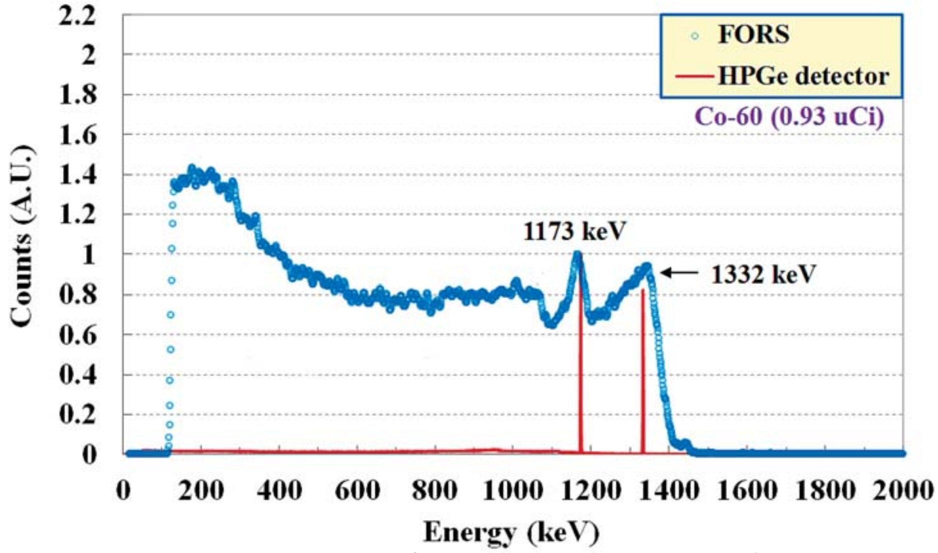
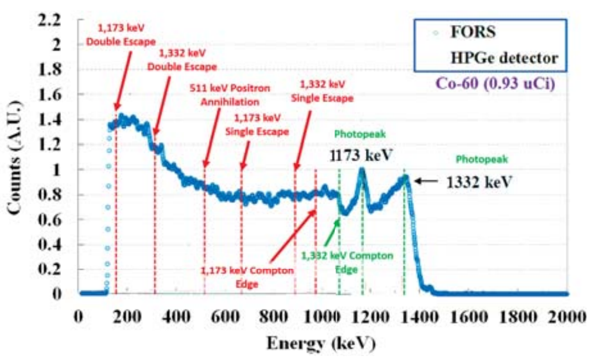
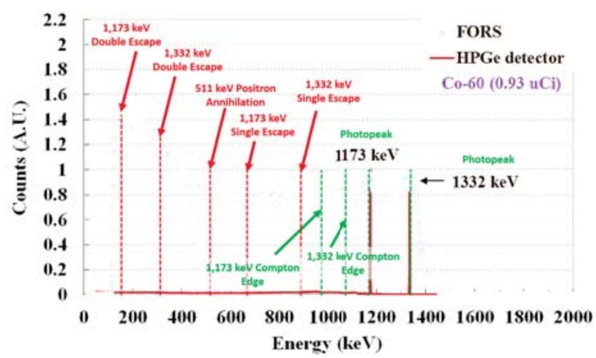


Image courtesy of *Sensors*—Open Access Journal.

2.1 (5 points) Label every observed feature of physical significance on the two spectra (separately for FORS and HPGe). Redraw the spectra separately in your test booklet for ease of reading.

Answer: There are four features of interest actually visible in the HPGe spectrum, and only three visible in the FORS spectrum. Each has two photopeaks, one at 1,173 keV and one at 1,332 keV, and each photopeak should have a corresponding Compton edge visible. However, the lower energy Compton edge is not visible in the FORS spectrum, probably due to its poor energy resolution and higher noise. It's too hard to discern whether others are noise or signal, so they will be addressed in the next part. The two Compton edges are visible in the HPGe spectrum. The two annotated spectra are shown below, with clearly observable features in green, and features which should have been there (Problem 2.2) in red.





2.2 (10 points) Denote the location of every physically significant peak which should be present from each spectrum, but was not observed. Why was each not observed?

Answer: The reasons that each red peak was not observed are different for each detector. For the FORS detector, which uses a tiny piece of plastic/glass, the energy resolution is terrible. This is clearly evident from the large spread on the photopeaks, and the very, very high background even between photopeaks. Therefore, even if other peaks were to be observed, they likely aren't discernible due to poor energy resolution and noise.

For the HPGe detector, this appears to be an extremely high resolution, but also extremely *large* semiconductor detector. Therefore, if the detector were large enough, almost none of the annihilation photons created from pair production would escape, giving no single- or double-escape peaks. In addition, we don't know what materials surround the detector. If it was just in air or something, or the surrounding material was low-Z, then there would be very little matter near the detector to undergo its own pair production, leading to relatively few 511 keV photons produced outside the detector.

2.3 (11 points) Develop simplified expressions for the locations in keV for physically significant features that are or should be present on the spectra, as functions of the observed photopeak energies E_1 (1,173 keV) and E_2 (1,332 keV). You do not need to calculate the actual locations, just make expressions that would evaluate to the correct answer.

- E_1 and E_2 photopeaks: $E_\tau = E_i - \phi_0$, where ϕ_0 is the work function of the detector material.
- E_1 and E_2 Compton edges, where the energy is that of the Compton backscattered electron at $\theta = \pi$:
  $T_i = E_i \frac{\alpha_i \left(1 - \cancel{cos(\theta)}^{-1}\right)}{1+\alpha_i\left(1-\cancel{cos(\theta)}^{-1}\right)} = E_i \frac{2\alpha_i}{1+2\alpha_i}; \qquad \alpha_i = \frac{E_i}{m_e c^2} = \frac{E_i}{511\,keV}$
- E_1 and E_2 single escape peaks: $E_{\kappa 1} = E_i - 511\,\text{keV}$
- E_1 and E_2 double escape peaks: $E_{\kappa 2} = E_i - 1,022\,\text{keV}$
- *One* 511 keV annihilation peak for all pair production annihilation photons created outside the detector

3 (26 points) Teetering with Criticality - The Demon Core

The Demon Core was a barely sub-critical sphere of plutonium that claimed two lives in two different experimental mistakes. One time, a tungsten carbide reflector brick fell on top of the barely sub-critical Demon Core, reflecting the neutrons back in and causing the core to briefly go supercritical.

3.1 (6 points) Develop an expression for the critical radius of a bare sphere of plutonium, in terms of its material properties and nuclear data.

Answer: $R = \frac{1}{\pi}\sqrt{\frac{\overline{D}}{\nu\overline{\Sigma}_f - \overline{\Sigma}_a}}$

Explanation: This expression *in particular* I would expect you to quickly be able to re-derive, if you didn't remember it or write it on your note sheet. Let's assume you didn't, and we'll re-derive it together. We write the criticality of the reactor (k_{eff}) as a ratio of gains to losses. Every gain/loss rate must be expressed as a reaction rate, in the form of $c\Sigma\Phi$, where c is whatever unitless constant is necessary, $\overline{\Sigma}$ is the appropriate *flux-averaged* macroscopic cross section, and Φ is the one-group neutron flux. The gain terms in the one-group equation consist only of fission $\left(\nu\overline{\Sigma}_f\Phi\right)$. The loss terms consist of absorption $\left(\overline{\Sigma}_a\Phi\right)$ and leakage, which takes a slightly different form. Recall that our one-group diffusion approximation yields a term which looks like $\Phi = -D\nabla^2\Phi$. The solution to this differential equation takes the form of $\frac{-\nabla^2\Phi}{\Phi} = Constants$. The simplest form of this equation in Cartesian coordinates is $\Phi = A\,sin\,(Bx) + C\,cos\,(Bx)$, but the cosine term disappears due to symmetry concerns. That yields a solution which looks like $\frac{-\nabla^2\Phi}{\Phi} = B^2$. We denote this constant B^2 as the *geometric buckling*, and for a sphere it is defined as $\left(\frac{\pi}{R_{ex}}\right)^2$, where $R_{ex} = R + 2D$. For reference, the diffusion coefficient is defined as $D = \frac{1}{3(\Sigma_t - \bar{\mu}_0\Sigma_s)}$, and μ_0 is the average cosine of the scattering angle, approximated as $\mu_0 \approx \frac{2}{3A}$.

This yields a full gains/losses ratio as:

$$k_{eff} = 1 = \frac{\nu\overline{\Sigma}_f\not{\Phi}}{\overline{\Sigma}_a\not{\Phi} + \overline{D}B_g^2\not{\Phi}} = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \overline{D}B_g^2} = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \overline{D}\left(\frac{\pi}{R+2D}\right)^2} \approx \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \overline{D}\left(\frac{\pi}{R}\right)^2} \tag{10}$$

where we have set $k_{eff} = 1$ for a critical system, and we have neglected the extrapolation distance because plutonium has a pretty hefty fission cross section, so its diffusion coefficient will be quite small. Now we just isolate R to get the answer:

$$\overline{\Sigma}_a + \overline{D}\left(\frac{\pi}{R}\right)^2 = \nu\overline{\Sigma}_f \tag{11}$$

$$\left(\frac{\pi}{R}\right)^2 = \frac{\nu\overline{\Sigma}_f - \overline{\Sigma}_a}{\overline{D}} \tag{12}$$

$$\frac{\pi}{R} = \sqrt{\frac{\nu\overline{\Sigma}_f - \overline{\Sigma}_a}{\overline{D}}} \tag{13}$$

$$R = \frac{1}{\pi}\sqrt{\frac{\overline{D}}{\nu\overline{\Sigma}_f - \overline{\Sigma}_a}} \tag{14}$$

3.2 (20 points) Explain what happened to the criticality of the sphere during each of the following events in the demon core. Using the one-group criticality relation, explain physically what happened to the parameters in each step, and answer each question. <u>Follow this example:</u>

$$Control\,rods\,lowered\,into\,reactor \qquad k_{eff}\downarrow\downarrow = \frac{A + B\downarrow}{C\uparrow + D\downarrow} \tag{15}$$

Common Answer: Note that the one-group criticality relation is as we defined above:

$$k_{eff} = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \overline{D}B_g^2} \tag{16}$$

Now we just have to describe what is happening at each stage with arrows.

3.2.1 The Demon Core was at rest, uncovered. Write the *criticality inequality*, and state physically why the sphere was sub-critical.

Answer: You can either say what k_{eff} should be with an inequality:

$$k_{eff} = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \overline{D}B_g^2} < 1 \tag{17}$$

Or you can relate the pure materials and geometry of the system, noting which buckling (materials or geometry) is bigger:

$$\nu\overline{\Sigma}_f < \overline{\Sigma}_a + \overline{D}B_g^2 \tag{18}$$

$$\frac{\nu\overline{\Sigma}_f - \overline{\Sigma}_a}{\overline{D}} = B_m^2 < B_g^2 \tag{19}$$

Either way, it says that the geometry of the given sub-critical system is too small to account for the mix of materials used to make it. There's too much absorption and leakage going on to support criticality.

3.2.2 The tungsten carbide brick was dropped onto the Demon Core. (Write an expression for the total macroscopic absorption cross section of tungsten carbide (WC) in terms of its constituent elements and microscopic cross sections)

Answer: Adding the tungsten carbide acts as a reflector, stopping many of the leaking neutrons from escaping. That only affects the leakage term, decreasing it, which increases k_{eff}:

$$k_{eff} \uparrow = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \ \overline{D}B_g^2 \ \downarrow} \tag{20}$$

The expression for the total macroscopic absorption cross section of tungsten carbide, assuming there is only one isotope of each element that matters, is an atom-fraction-weighted sum of their number densities and microscopic cross sections. Tungsten carbide is easy, since both W and C are 50% atom fraction, and therefore each of their number densities is the same as the tungsten carbide molecule itself:

$$\overline{\Sigma}_{a_{WC}} = N_W\overline{\sigma}_{a_W} + N_C\overline{\sigma}_{a_C} = N_{WC}\left(\overline{\sigma}_{a_W} + \overline{\sigma}_{a_C}\right) \tag{21}$$

where we define the flux-averaged cross section as:

$$\overline{\sigma} = \frac{\int \sigma(E)\,\Phi(E)\,dE}{\int \Phi(E)\,dE} \tag{22}$$

3.2.3 The Demon Core went prompt-supercritical, releasing a huge amount of power. (***What does prompt-supercritical mean?***)

Answer: After the core goes prompt-supercritical and releases tons of power, it will get very hot from a temperature point of view. This will cause all the *flux-averaged* cross sections to increase by Doppler broadening (temperature feedback), with the net effect of letting neutrons travel farther and increasing leakage. We neglect thermal expansion making the sphere physically bigger:

$$k_{eff} \downarrow = \frac{\nu\overline{\Sigma}_f \uparrow}{\overline{\Sigma}_a \uparrow + \ \overline{D}B_g^2 \ \uparrow} \tag{23}$$

Prompt-supercritical means that an amount of reactivity (fraction of k_{eff}) was added that is greater than the fraction of delayed neutrons (β), so there is no slow feedback of neutrons to be had. The power increases exponentially with time constants in the millisecond range.

3.2.4 Fission products built up in the Demon Core. (This doesn't mean that it went subcritical just yet...)

Answer: Fission products are usually very strong absorbers, which both increases absorption greatly, slightly decreases the fissile number density (and therefore its macroscopic cross section) and lowers the diffusion coefficient by increasing absorption:

$$k_{eff}\downarrow = \frac{\nu\overline{\Sigma}_f\downarrow}{\overline{\Sigma}_a\upuparrows + \overline{D}\downarrow B_g^2} \tag{24}$$

3.2.5 The tungsten carbide brick was knocked off the Demon Core, ending the criticality excursion.

Answer: This ended the extra reflection, allowing the neutrons to leak back out:

$$k_{eff}\downarrow = \frac{\nu\overline{\Sigma}_f}{\overline{\Sigma}_a + \left(\overline{D}B_g^2\right)\uparrow} \tag{25}$$

References

[1] W. J. Yoo et al. "Development of a Small-Sized, Flexible, and Insertable Fiber-Optic Radiation Sensor for Gamma-Ray Spectroscopy." *Sensors*, 15(9):21265 (2015).

Useful Formulas

$$T = E\frac{\alpha\left(1-cos\left(\theta\right)\right)}{1+\alpha\left(1-cos\left(\theta\right)\right)} \qquad \frac{\omega}{\omega'} = 1+\alpha\left(1-cos\left(\theta\right)\right) \qquad \alpha = \frac{E}{m_e c^2} \tag{26}$$

$$\frac{d\sigma}{d\theta} = \frac{k_0^2 e^4}{2m_e^2 c^4}\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'}+\frac{\nu'}{\nu} - sin^2\theta\right) \tag{27}$$

$$\Delta\lambda = \frac{h}{m_e c}\left(1-cos\left(\theta\right)\right) \qquad cot\left(\phi\right) = \left(1+\alpha\right)tan\left(\frac{\theta}{2}\right) \tag{28}$$

$$\sigma_C \propto \frac{Z}{E} \qquad \sigma_\tau \propto \frac{Z^5}{E^{7/2}} \qquad \sigma_\kappa \propto Z^2 ln\left(\frac{2E}{m_e c^2}\right) \tag{29}$$

$$Q = T_3\left(1+\frac{M_3}{M_4}\right) - T_1\left(1-\frac{M_1}{M_4}\right) - \frac{2}{M_4}\sqrt{T_1 T_3 M_1 M_3}cos\left(\theta\right) \tag{30}$$