Lecture 4: Stabilizers

5 qubit code

$$|0_L\rangle = |00000\rangle + |11000\rangle + |01100\rangle + |00110\rangle + |00011\rangle + |10001\rangle - |10100\rangle - |01010\rangle - |00101\rangle - |10010\rangle - |01001\rangle - |11110\rangle - |01111\rangle - |10111\rangle - |11011\rangle - |11101\rangle$$

$$|1_L\rangle = \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \; |0_L\rangle$$

$$= |11111\rangle + \text{c.s.}\{|00111\rangle\} + \text{c.s.}\{|10101\rangle\} + \text{c.s.}\{|00001\rangle\}$$

Stabilizer Construction

Group of tensor products of Pauli Matrices

e.g. $\sigma_z^{(1)} \otimes \sigma_x^{(2)} \otimes \sigma_x^{(3)} \otimes \sigma_z^{(4)} \otimes I^{(5)} = ZXXZI$

Take 4 commuting elts ($\Rightarrow$ simult diagonalizable)

$g_1 = ZXXZI$
$g_2 = IZXXZ$
$g_3 = ZIZXX$
$g_4 = XZIZX$

(Remaining cyclic perm: $XXZIZ = I$)

Look at simult eigenspaces w/ eigenvalues

$g_1 = +1$
$g_2 = -1$
$g_3 = +1$
$g_4 = -1$

$\Rightarrow$ 16 possible sets of eigvals

16 2-dim eigspaces

Eigspaces are QEC codes! Dude!

e.g. $ZXXZI\,|11000\rangle = -|10100\rangle$ (both kets are terms in $|0_L\rangle$)

Why do we get 16 eigspaces all of same dim?

Let h be a tensor prod of Paulis:

$h = XIIIII$ $\qquad\qquad$ $|\psi\rangle \in$ code subspace

$$g_1 h|\psi\rangle = -hg_1|\psi\rangle = h|\psi\rangle$$
$$g_2 h|\psi\rangle = h|\psi\rangle$$
$$g_3 h|\psi\rangle = -h|\psi\rangle$$
$$g_4 h|\psi\rangle = h|\psi\rangle$$

So h takes $|\psi\rangle$ to subspace w/ $\{-1,+1,-1,+1\} \Rightarrow$ dim 4

Do algebra for other h's to confirm rest for others

Why does this correct one error?

Error e is a tensor prod of Pauli matrices

$|\psi\rangle \longrightarrow e|\psi\rangle \in$ another eigspace

Must figure out how to get $e|\psi\rangle$ back to original eigspace

Let $e_1|\psi\rangle = e_2|\tilde{\psi}\rangle \Rightarrow e_1 e_2|\tilde{\psi}\rangle = |\psi\rangle$

Then $e_1 e_2 \{\text{code subspace}\} =$ code subspace

$$e_1 e_2|\tilde{\psi}\rangle = e_1 e_2 g_i|\tilde{\psi}\rangle \quad \forall\, |\tilde{\psi}\rangle \in C$$
$$= g_i e_1 e_2 |\tilde{\psi}\rangle$$

$\Rightarrow g_i$ commutes w/ e_1, e_2

let $h_1 = XXXXX$
$h_2 = ZZZZZ$

h_1, h_2 commute w/ g_1, g_2, g_3, g_4

$h_1 |0_L\rangle = |1_L\rangle$
$h_2 |0_L\rangle = |0_L\rangle$
$h_1 |1_L\rangle = -|1_L\rangle$
$h_2 |1_L\rangle =$

$h_1 g_1 = YIIYX$ commutes w/ g_1, g_2, g_3, g_4
$\Rightarrow$ can't distinguish $Y^{(1)}|\psi\rangle$ from $Y^{(4)} \otimes X^{(5)} |\psi\rangle$

$\mathcal{H} = \{ g \in \otimes\sigma \mid g g_i = g_i g \;\; \forall g_i \}$

min wt $(\mathcal{H})$ = min distance of code

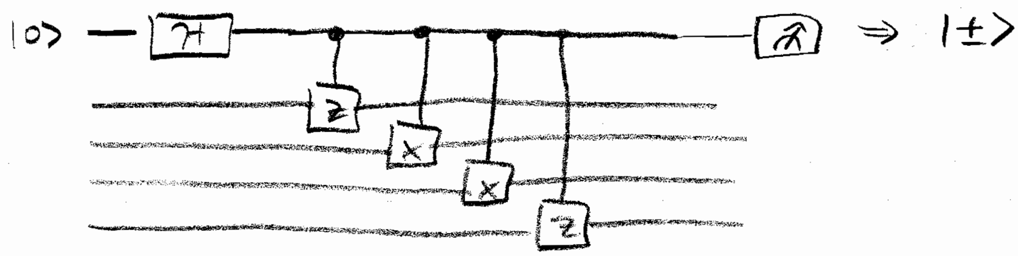
☆ Can correct $\frac{1}{2}(\text{min wt}(\mathcal{H}) - 1)$ errors

let $|\psi_{err}\rangle = Y^{(1)}|\psi\rangle$

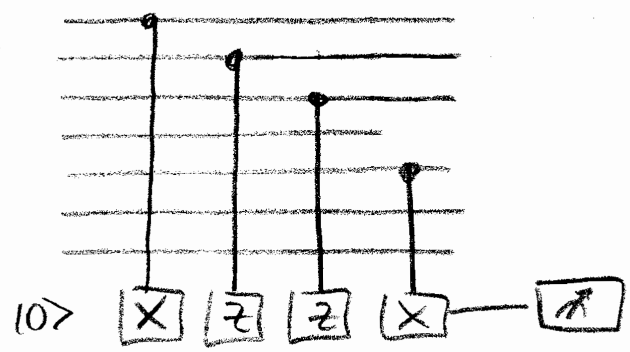
$g_1 |\psi_{err}\rangle = - |\psi_{err}\rangle$
$g_2 |\psi_{err}\rangle = + |\psi_{err}\rangle$
$g_3 |\psi_{err}\rangle = - |\psi_{err}\rangle$
$g_4 |\psi_{err}\rangle = - |\psi_{err}\rangle$

$\Rightarrow$ Syndrome: $\begin{bmatrix} -1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

How do we measure the syndrome?



Similar to CSS code:



(possibly miscoped)

Recall

$$
\begin{array}{ll}
\begin{matrix} Z & Z & Z & I & Z & I & I \\ I & Z & Z & Z & I & Z & I \\ I & I & Z & Z & Z & I & Z \end{matrix} & \leftarrow C_2^{\perp} \\
\begin{matrix} X & X & X & I & X & I & I \\ I & X & X & X & I & X & I \\ I & I & X & X & X & I & X \end{matrix} & \leftarrow C_1
\end{array}
$$

$$e_1|\psi\rangle = e_2|\tilde{\psi}\rangle$$

$$e_1|\psi\rangle = e_2|\psi\rangle \Rightarrow e_1 e_2|\psi\rangle \quad \forall |\psi\rangle \in \text{code space}$$

$$e_1 e_2 \in \{g_1, g_2, g_3, g_4\}$$

$\mathcal{H} = \{ h \mid g_i h = h g_i \ \forall g_i \}$

$d = \min \text{wt} \{ x \in \mathcal{H} - G \}$

Code corrects $\frac{1}{2}(d-1)$ errors

$[[n, k, d]]$ code

$n = \log_2(\dim)$

$\log_2\left(\frac{2^n}{2^{|g_i|}}\right) = n - g_i$

So 1st code example today is $[[5, 1, 3]]$

Stabilizers in Classical Codes

<u>GF(4)</u> has elts $0, 1, \omega, \bar{\omega}$

with: $\omega \cdot \bar{\omega} = 1$

$1 + \omega = \bar{\omega}$

$\omega^2 = \bar{\omega}$

<u>Trace</u>: $\text{Tr}\ 0 = 0$
$\text{Tr}\ 1 = 0$
$\text{Tr}\ \omega = 1$
$\text{Tr}\ \bar{\omega} = 1$

<u>Inner Product</u> $(a, b) = \text{Tr}\ \bar{a} b$

Let's map
$X \to \omega$
$Y \to \bar{\omega}$
$Z \to 1$
$I \to 0$

$g_1 = 1 \omega \omega 1 0$
$g_2 = 0 1 \omega \omega 1$
$g_3 = 1 0 1 \omega \omega$
$g_4 = \omega 1 0 1 \omega$

commuting $(g_i, g_j) = 0$

e.g. $(g_1, g_2) = \text{Tr}(0 + \bar{\omega} + 1 + \omega + 0) = 0$

Quantum stabilizer codes

$\leftrightarrow$ additive weakly self-dual codes over $GF(4)$

additive: $g_1 + g_2 = 1\,\bar{\omega}\,0\,\bar{\omega}\,1$

$\bar{\omega} g_4 = 1\,\bar{\omega}\,0\,\bar{\omega}\,1$

$\boxed{\text{Hexacode}}$ – a linear code over $GF(4)$

$$\begin{matrix} 1 & \omega & \omega & 1 & 0 & 0 \\ 0 & 1 & \omega & \omega & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$\Rightarrow$ quantum code $(6, 0, 4)$

Take codewords of hexacode w/ last letter $= 0$

$\Rightarrow$ get 5-qubit code $(5, 1, 3)$

Can delete another one to get $(4, 2, 2)$

but not so powerful

$\boxed{\text{Gilbert-Varshamov bound for } GF(4) \text{ code}}$

$[[n, k, d]]$

Rate $R = k/n$

$\delta = d/n$ twice # correctable errors

G-V bound says asymptotically, approach

$$1 - R \sim \delta \log_2 3 + \mathcal{H}_2(\delta)$$

There is an additive self-dual
$[[12, 0, 6]]$ code
Get by looking @ cyclic shifts of
$\omega 1 0 1 0 0 1 0 0 1 0 1$
$\Rightarrow$ $[[11, 1, 5]]$ code