Lecture 16: Quantum Channels III

Quantum channel Φ

Define $\chi(\Phi) = \underbrace{H\left(\Phi\left(\sum p_i \rho_i\right)\right)}_{\text{entropy of average output}} - \underbrace{\sum p_i H(\Phi(\rho_i))}_{\text{average of output entropy}}$

Take max over all p_i, ρ_i

<u>Accessible info:</u>

$$\text{Message } M \left\{ \begin{array}{l} v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \right. \xrightarrow{\Phi} \begin{array}{l} \Phi(v_1) \\ \Phi(v_2) \\ \vdots \\ \Phi(v_n) \end{array} \quad \overset{\text{Measure:}}{\left. \begin{array}{l} E_1 \\ E_2 \\ \vdots \\ E_n \end{array} \right\}} \Rightarrow M$$

Strictly less than

$$\left\{ \begin{array}{l} v_1 \\ v_2 \\ \vdots \\ v_n \end{array} \right. \longrightarrow \left. \begin{array}{l} \Phi(v_1) \\ \Phi(v_2) \\ \vdots \\ \Phi(v_n) \end{array} \right\} \text{joint measurement}$$

$(\max \chi)$

$\chi >$ A.I. unless $\Phi(v_i)\Phi(v_j) = \Phi(v_j)\Phi(v_i) \ \forall i,j$

<u>Is it better to use entangled input?</u> (open Q)

$$\lim_{n\to\infty} \max \frac{\chi(\Phi^{\otimes n})}{n} \geq \max \chi(\Phi)$$

<u>Additivity Q</u> (Open Q): Is $\max \chi(\Phi_1) + \max \chi(\Phi_2) \stackrel{?}{=} \max \chi(\Phi_1 \otimes \Phi$
($\leq$ easy to show.) ($\geq$ not known)

Equivilent Q to additivity of min entropy output

Is $\min \mathcal{H}(\Phi(\rho))$ additive?

$$\min \mathcal{H}(\Phi) + \min \mathcal{H}(\Phi_2) \overset{?}{\underset{\geq}{=}} \min \mathcal{H}(\Phi_1 \otimes \Phi_2)$$

($\geq$ easy) ($\leq$ unknown)

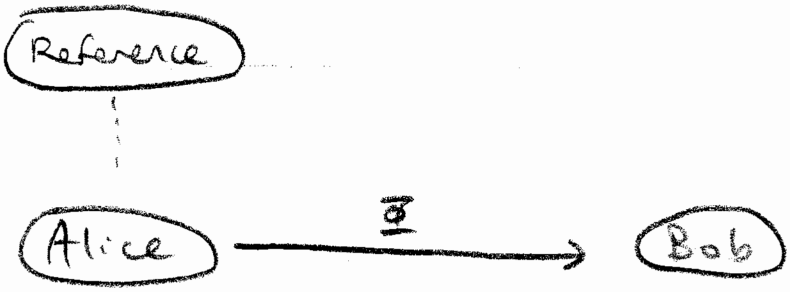
What else might improve quantum capacity?

Ideas:
- Classical feedback from Bob to Alice
  - yes, if additivity assumed
- Entanglement pre-shared btwn Alice & Bob
  - yes (nice formula)

$$C_{op} = \max_A \mathcal{H}(B) - \mathcal{H}(B|A) \xrightarrow{\text{generalize}} \chi$$

$$= \max_A \mathcal{H}(B) + \mathcal{H}(B) - \mathcal{H}(A,B) \xrightarrow{\text{generalize}} \text{entanglement assisted capacity}$$

(not equal in general)



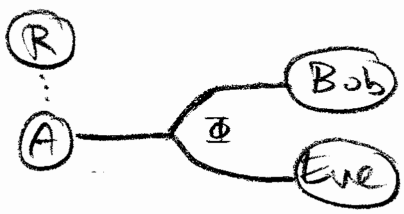
Alice inputs 1/2 of a pure entangled state btwn A & R

$\mathrm{Tr}_R \Psi_\rho = \rho$

Entanglement-assisted capacity is:

$$\max_\rho \mathcal{H}(\rho) + \mathcal{H}(\Phi(\rho)) - \mathcal{H}(I \otimes \Phi(\Psi_\rho))$$

Say



$\mathcal{H}(\rho_{ABE}) = 0$

$$\begin{aligned} E\text{-}A.C. &= \mathcal{H}(\rho_R) + \mathcal{H}(\rho_B) - \mathcal{H}(\rho_{RB}) \\ &= \mathcal{H}(\rho_R) + \mathcal{H}(\rho_B) - \mathcal{H}(\rho_E) \end{aligned}$$

If $\rho = I/2$ for qubits

Protocol: A & B have ∞ supply of EPR pairs

A takes her half

Applies id, σ_x, σ_y, σ_z w/prob 1/4 & sticks into channel

4 signal states: $\Phi \otimes I(\Psi)$

$\Phi \otimes I(\sigma_y \Psi \sigma_y)$

$\Phi \otimes I(\sigma_x \Psi \sigma_x)$

$\Phi \otimes I(\sigma_z \Psi \sigma_z)$

Apply HSW Thm: $b \in \{x, y, z, id\}$

Gives $\mathcal{H}\left(\frac{1}{4} \sum \Phi \otimes I(\sigma_b \Psi \sigma_b)\right) - \frac{1}{4} \sum \mathcal{H}\left(\Phi \otimes I(\sigma_b \Psi \sigma_b)\right)$

$= \mathcal{H}\left(\Phi \otimes I(I/2 \otimes I/2)\right)$ $\mathcal{H}(\Phi(\rho)) + \mathcal{H}(\rho)$

Proved for $\rho = P/k$ (P = projection)

But suppose P <u>not</u> a projection matrix?

Use result for $\Phi^{\otimes n}$ & $\rho = \Pi_{T_{\rho^{\otimes n}}}$

(where $\Pi_{T_{\rho^{\otimes n}}}$ = projection matrix onto typical subspace)

Need $\mathcal{H}(\Phi^{\otimes n}(T_{\rho^{\otimes n}})) \approx n \mathcal{H}(\Phi(\rho))$ (Provable)

Quantum Capacity of Channel

$$|\phi\rangle \xrightarrow{\text{unitary}} |\nu\rangle \xrightarrow{\Phi^{\otimes n}} \Phi^{\otimes n}(|\nu\rangle\langle\nu|)$$

$$\mathbb{C}^{2^{\otimes d}} \qquad \mathbb{C}^{2^{\otimes n}}$$

Bob decodes to get $\rho \in \mathbb{C}^{2^{\otimes d}}$

Want $\langle\phi|\rho|\phi\rangle = 1-\varepsilon$

- for average $|\phi\rangle \in \mathbb{C}^{2^{\otimes d}}$
- or for worst-case $|\phi\rangle \in \mathbb{C}^{2^{\otimes d}}$
- or for $|\phi\rangle$ maximally entangled btwn $\mathbb{C}^{2^{\otimes d}}$ & R

some reference system

Which is right def? <u>All</u>.

<u>Coherent information:</u>

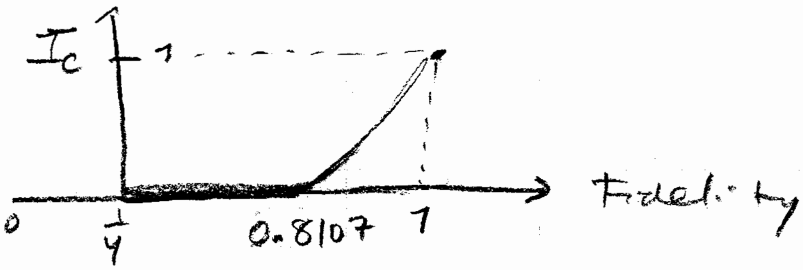
$$\max I_c = \max_{\rho} \mathcal{H}(\Phi(\rho)) - \mathcal{H}(\Phi \otimes I(\psi_\rho))$$

<u>Proof sketch:</u> Choose random subspace of $T\rho^{\otimes n}$ of right dim to achieve I_c

$$Q := \lim_{n\to\infty} \frac{1}{n} \max_{\rho} I_c(\Phi^{\otimes n})$$

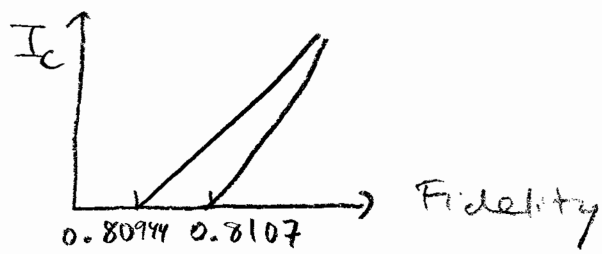
Sometimes we need limit

For depolarizing channel,



maximized when $\rho = I/2$

For $n = 5$ or 25,



<u>What is improvement?</u>

Use $n = 3$ $\quad \rho = \frac{1}{2}\left(|000\rangle\langle 000| + |111\rangle\langle 111| \right)$

(coding subspace of 3-repition code)

Can do better using 5-repition code
(But do <u>worse</u> using 7-repetition code)
Can do even better using coding subspace of 9-qubit code
or w/ 25-qubit Shor-Bacon code
though gets really hard (impossible?) to compute

Consider noisy channel Φ

Want to: a) send quantum bits

b) send classical bits

c) use entanglement

d) use classical communication

etc

Suppose we have Φ.

Want to send αn qubits

βn cbits

using γn entanglements (max)

What is min # of Φ? $\delta n + o(n)$

<u>Resources</u>: $[c \to c]$ one classical bit transmitted

$[q \to q]$ one quantum bit transmitted

$[qq]$ EPR pair

<u>Teleportation</u>: $2[c \to c] + [qq] = [q \to q]$

<u>Super-dense coding</u>: $[q \to q] + [qq] \geq 2[c \to c]$

$[q \to q] \geq [qq]$

<u>HSW Thm</u>: $\langle \Phi : \rho^A \rangle \geq \chi(\Phi : \rho^A) \cdot [c \to c]$

quantum channel + Alice's input

Father Protocol:

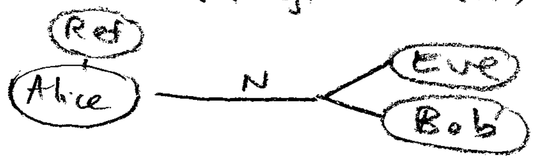
channel

Noisy channel

$$\boxed{\langle N \rangle + \tfrac{1}{2} I(R,E)[qq] \geq \tfrac{1}{2} I(R,E)[q \to q]}$$

where I = quantum mutual info

$$I(x:y) = H(x) + H(y) - H(xy)$$



$$H(R)[qq] + \langle N \rangle = \tfrac{1}{2} I(R,B)[qq] + \tfrac{1}{2} I(R,E)[qq] + \langle N$$

$$\geq \tfrac{1}{2} I(R,B)[qq] + \tfrac{1}{2} I(R,B)[q \to q]$$

$$\geq I(R,B)[c \to c]$$

entanglement

entanglement

$$\boxed{\langle \rho \rangle + \tfrac{1}{2} I(A,E)[q \to q] \geq \tfrac{1}{2} I(A,B)[qq]}$$