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4/6/06 Channel Capacities - II ('Shor')

$$\chi(p_i, |v_i\rangle) = H\left(\sum_i p_i |v_i\rangle\langle v_i|\right).$$

Allowed to send $\rho_1, \rho_2, \dots \rho_k$

Capacity is $\max_{\{p_i, \hat{\rho}_i\}} \chi = \max_{\{p_i, \rho_i\}} H\left(\sum_i p_i \hat{\rho}_i\right) - \sum_i p_i H(\hat{\rho}_i)$

block length: n

capacity: C

Pick $2^{n(C-\epsilon)}$ random codewords

w/ letters chosen w/ prob maximizing $H(B) - H(B/A)$

- Alice sends Bob codeword.
- Bob gets codeword with noise
- Finds the codeword most likely to have been the input works.

as $n \to \infty$, $\epsilon \to 0$.

How about quantum case?

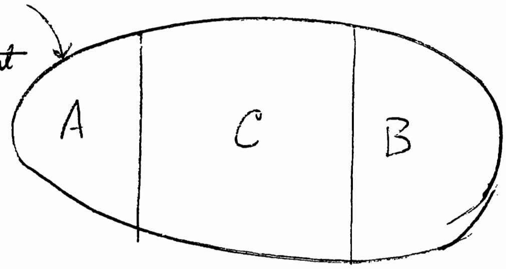
Upper bound:

Alice sends $\hat{\rho}_i$ to Bob

Will show that for single-state decoding,

Shannon information provided by any measurement of Bob's $<$ Holevo information χ

Alice's record of i (she sent ρ_i)



$$|i\rangle \longrightarrow \hat{\rho}_i \xrightarrow[\text{measurement}]{\text{Bob's}} |b_j\rangle$$

basis-states of measurement telling Bob the meas. result.

There could be a residual state $|\psi_i\rangle$ after Bob's measurement.

$I(A;B)$ Shannon-capacity

$$= H(A) + H(B) - H(A,B).$$

Holevo quantity

$$H\Big(\sum_i p_i \hat{\rho}_i\Big) - \sum_i p_i\, H(\hat{\rho}_i)$$

$H(\rho_{BC})$ $\qquad$ $H(\rho_{ABC}) - H(\rho_A)$

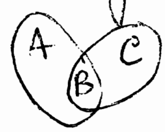
<u>We want</u> $H(A) + H(B) - H(AB)$

or, $H(\rho_A) + H(\rho_B) - H(\rho_{AB})$ ... just change of notation.

$$\leq H(\rho_{ABC}) + H(\rho_{BC}) + H(\rho_A)$$

$$0 \leq -H(\rho_{ABC}) + H(\rho_{AB}) + H(\rho_{BC}) - H(\rho_B)$$

Strong subadditivity of quantum entropy



Why is $\sum_i p_i H(\hat{\rho}_i) = H(\rho_{ABC}) - H(\rho_A)$

A is classical. $1, 2, \ldots k$

Alice has density-matrix:

$$\begin{pmatrix} p_1\hat{\rho}_1 & & & & 0 \\ & p_2\hat{\rho}_2 & & & \\ & & p_3\hat{\rho}_3 & & \\ & 0 & & \ddots & \\ & & & & p_k\hat{\rho}_k \end{pmatrix}$$

Entropy is $\sum_{i,j} -(p_i \lambda_{ij}) \log (p_i \lambda_{ij})$

$$= \sum_{i,j} -p_i \lambda_{ij} \log p_i - \sum_i p_i \lambda_{ij} \log \lambda_{ij}$$

$$= \sum_i -p_i \log p_i - \sum p_i H(\hat{\rho}_i)$$

Proof ingredients:
- random coding
- typical subspaces
- pretty-good measurement (SRM: Square-root-meas.)

We have codewords $|v_i\rangle$ which we use with associated probability p_i

Density matrix: $\hat{\rho} = \sum_i p_i |v_i\rangle\langle v_i|$

$$= \sum_i \lambda_i |\tilde{v}_i\rangle\langle \tilde{v}_i|$$

<u>Typical subspace of $\mathcal{H}^{\otimes n}$</u>

↳ is spanned by typical sequences of $|\tilde{v}_i\rangle$: $\{|v_i\rangle$ that appear $\approx \lambda \cdot n$ times

Random code

codeword $|\varphi_i\rangle$ is $|v_{i_1}\rangle \otimes |v_{i_2}\rangle \otimes \cdots \otimes |v_{i_n}\rangle$

$prob(v_i) = p_i$

Choose $N = 2^{n(H(p)-\epsilon)}$ codewords

Codewords $|\phi_1\rangle, \ldots, |\phi_N\rangle$

$$\Phi = \sum_{k=1}^{N} |\phi_k\rangle\langle\phi_k|$$

$$|\mu_k\rangle = \Phi^{-1/2}|\phi_k\rangle$$

$$\sum_{k=1}^{N} \underbrace{|\mu_k\rangle\langle\mu_k|}_{\text{(POVM elements)}} = \Phi^{-1/2}|\phi_k\rangle\langle\phi_k|\Phi^{1/2} = \mathbb{1}$$

S_{jk} matrix $= \langle\phi_j|\phi_k\rangle$

$$((\sqrt{S}))_{j,k} = \langle\mu_j|\phi_k\rangle$$
$$= \langle\phi_j|\Phi^{-1/2}|\phi_k\rangle \quad \ldots \text{ It's Hermitian.}$$

$$(\sqrt{S})^2_{j\ell} = \sum_k \langle\phi_j|\mu_k\rangle\langle\mu_k|\phi_\ell\rangle$$

$$= \langle\phi_j|\phi_\ell\rangle$$

Protol of encoding/decoding:—

- Alice sends $|0\rangle$
- Bob projects onto typical subspace.
- Bob applies SRM ($\sqrt{\ }$) measurement.

- P_E if Alice sent $|\phi_j\rangle$, is $1-(\langle\mu_j|\Pi_\Lambda|\phi_j\rangle)^2$
- Use $S_i = \Pi_\Lambda|\phi_i\rangle$

  projects onto typical subspace.

$$\longrightarrow 1-|\langle\mu_j|s_j\rangle|^2$$

- Average error

$$P_E = 1 - \sum_i \frac{1}{N}|\langle\mu_i|s_i\rangle|^2$$

$$= \frac{1}{N}\sum_i\left(1 - \langle\mu_i|s_i\rangle^2\right)$$

$$= \frac{1}{N}\sum_i\left(1-\langle\mu_i|s_i\rangle\right)\left(1+\langle\mu_i|s_i\rangle\right)$$

$$\leq \frac{2}{N}\left(1-\langle\mu_i|s_i\rangle\right)$$

$$\leq \frac{2}{N}\left(1-\sqrt{s_{ii}}\right)$$

$$\sqrt{s} \geq \frac{3}{2}s - \frac{1}{2}s^2$$

$$\leq \frac{2}{N}\sum_i\left(\frac{3}{2}S-\frac{1}{2}S^2\right)_{ii} = \frac{2}{N}\sum_i\left(1-\left[\frac{3S}{2}-\frac{1}{2}S^2\right]\right)_{ii}$$

$$\leq \frac{2}{N}\sum 1-\frac{3}{2}S_{ii} + \frac{1}{2}\sum_j S_{ij}S_{ji}$$

$$\leq \frac{2}{N}\sum\left(1-\frac{3}{2}n_i+\frac{1}{2}n_i^2\right)+\frac{1}{N}\sum_{j\neq i}S_{ij}S_{ji}$$

$$|S_k\rangle = \Pi_\Lambda|\phi_k\rangle$$

$$\langle s_k|S_k\rangle = \langle\phi_k|\Pi_\Lambda|\phi_k\rangle$$

$$E\langle S_k|S_k\rangle = 1-\epsilon$$

$$S_{ij}S_{ji} = \cancel{\langle S_i|\Pi_\Lambda|S_j\rangle}$$
$$\langle S_i|S_j\rangle\langle S_j|S_i\rangle$$
$$= E\langle\phi_i|\Pi_\Lambda|\phi_j\rangle\langle\phi_j|\Pi_\Lambda|\phi_i\rangle$$
$$= Tr\left(\Pi_\Lambda|\phi_j\rangle\langle\phi_j|\Pi_\Lambda|\phi_i\rangle\langle\phi_i|\right)$$
$$= Tr\left(\Pi_\Lambda\rho^{\otimes n}\Pi_\Lambda\rho^{\otimes n}\right)$$
$$= Tr\left(\Pi_\Lambda(\rho^{\otimes n})^2\Pi_\Lambda\right)$$

$\Pi_\Lambda\rho^{\otimes l}\Pi_\Lambda$ has $\approx 2^{nH(\hat\rho)}$ eigenvectors $\neq 0$

$|\phi_i\rangle = |v_{i_1}\rangle\otimes|v_{i_2}\rangle\otimes\cdots\otimes|v_{i_n}\rangle$

each of them has eigenvalue $\approx 2^{-n(H(\rho)+\epsilon n)}$

Square it: get $2^{nH(\hat\rho)}$ evectors w/ e.v. $2^{-nH(\hat\rho)}$

add $N=2^{nH(\hat\rho)-\epsilon n}$ of these up

—x—