Introduction to Numerical Analysis for Engineers

Roots of Nonlinear Equations

$$f(x) = 0$$

Example – Square root

$$x^2 - a = 0 \Rightarrow x = \sqrt{a}$$

Heron's Principle

$$x > 0$$

$$x^2 - a = 0 \Leftrightarrow x = \frac{a}{x}$$

Guess root

$$x_0 > \sqrt{a} \quad \Leftrightarrow \quad \frac{a}{x_0} < \sqrt{a}$$

$$x_0 < \sqrt{a} \quad \Leftrightarrow \quad \frac{a}{x_0} > \sqrt{a}$$

Mean is better guess

$$x_1 \quad = (x_0 + \frac{a}{x_0})/2$$

Iteration Formula

$$x_k \quad = (x_{k-1} + \frac{a}{x_{k-1}})/2$$

heron.m

```
a=2;
n=6;
g=2;
% Number of Digits
dig=5;
     sq(1)=g;
     for i=2:n
      sq(i)= 0.5*radd(sq(i-1),a/sq(i-1),dig);
     end
     '      i        value    '
     [ [1:n]' sq']
     hold off
     plot([0 n],[sqrt(a) sqrt(a)],'b')
     hold on
     plot(sq,'r')
     plot(a./sq,'r-.')
     plot((sq-sqrt(a))/sqrt(a),'g')
     grid on
```

i	value
1.0000	2.0000
2.0000	1.5000
3.0000	1.4167
4.0000	1.4143
5.0000	1.4143
6.0000	1.4143

Roots of Nonlinear Equations
Stop-criteria

Unrealistic stop-criteria

$$x_{k+1} \neq x_k$$

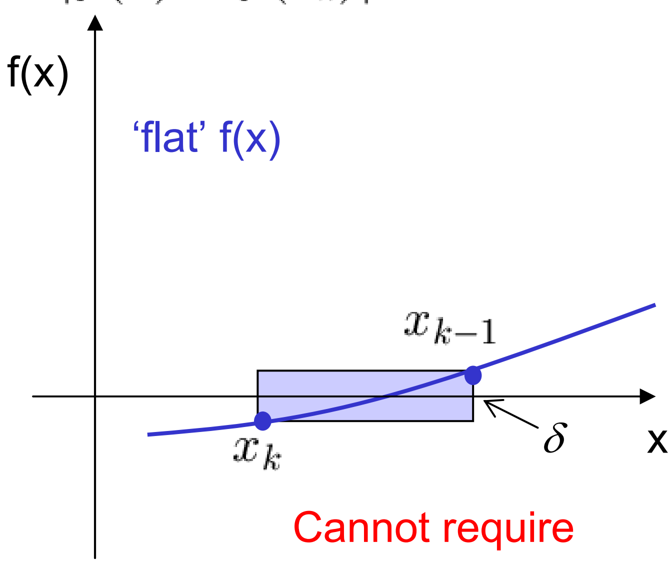
Realistic stop-criteria

$$|x_k - x_{k-1}| \;<\; \delta$$

$$|f(x) - f(x_k)| \;<\; \delta$$

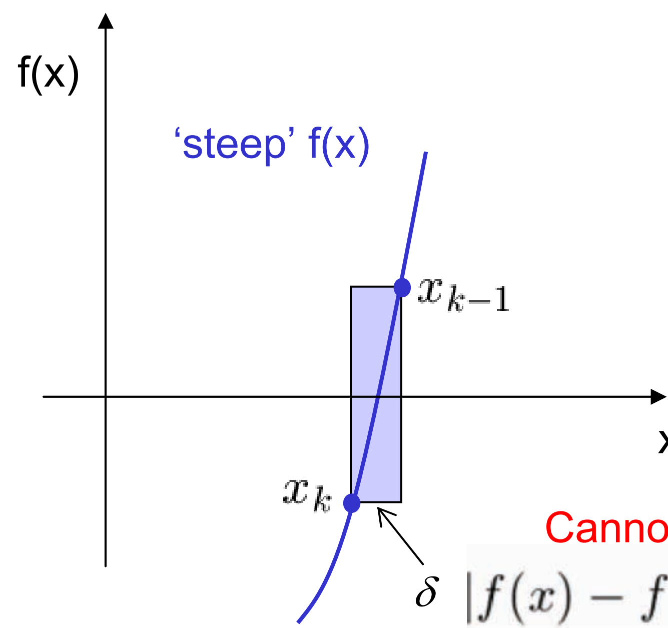
Machine Accuracy

$\boxed{\text{Use combination of the two criteria}}$



Cannot require

$$|x_k - x_{k-1}| \;<\; \delta$$



Cannot require

$$|f(x) - f(x_k)| \;<\; \delta$$

Roots of Nonlinear Equations
General Method

Non-linear Equation

$$f(x) = 0$$

Goal: Converging series

$$x_0, x_1, \ldots x_n \rightarrow x^e \;, \quad n \rightarrow \infty$$

Rewrite Problem

$$f(x) = 0 \Leftrightarrow g(x^e) = x^e$$

Example

$$g(x) = x + c \cdot f(x)$$

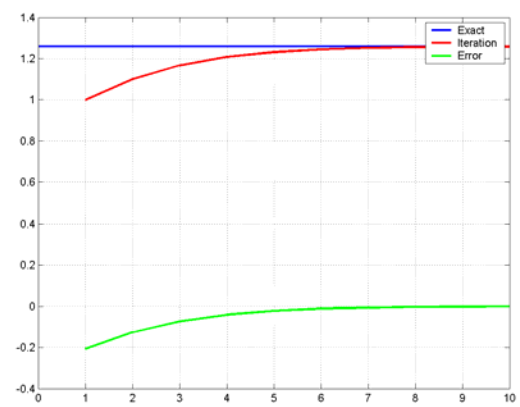
Iteration

$$x_n = g(x_{n-1})$$

Example: Cube root

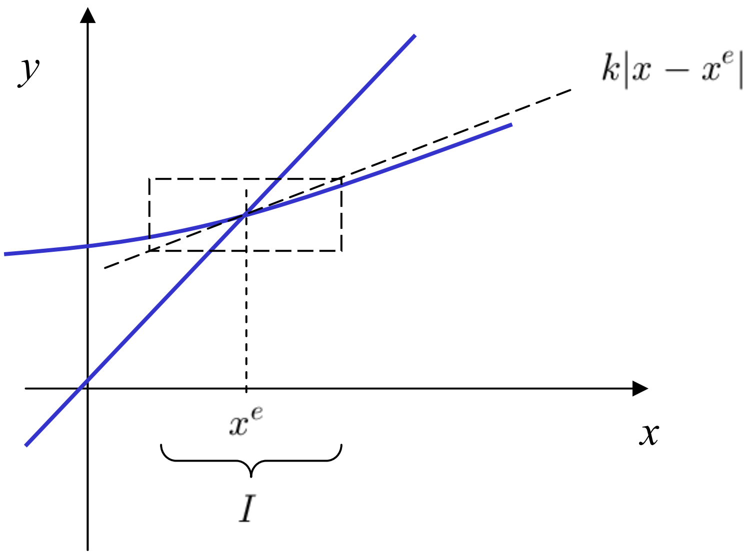
cube.m

```
% f(x) = x^3 - a = 0
% g(x) = x + C*(x^3 - a)
a=2;
n=10;
g=1.0;
C=-0.1;
     sq(1)=g;
     for i=2:n
      sq(i)= sq(i-1) + C*(sq(i-1)^3 -a);
     end
     hold off
     plot([0 n],[a^(1./3.) a^(1/3.)],'b')
     hold on
     plot(sq,'r')
     plot( (sq-a^(1./3.))/(a^(1./3.)),'g')
     grid on
```



Roots of Nonlinear Equations
General Method

Convergence



Define k such that if

$$x \in I$$

then

$$|g(x) - g(x^e)| = |g(x) - x^e| \leq k|x - x^e|$$

Convergence Criteria

$$x_{n-1} \in I \Rightarrow |x_n - x^e| = |g(x_{n-1}) - x^e| \leq k|x_{n-1} - x^e|$$

Apply successively

$$|x_n - x^e| \leq k^n|x_0 - x^e|$$

Convergence

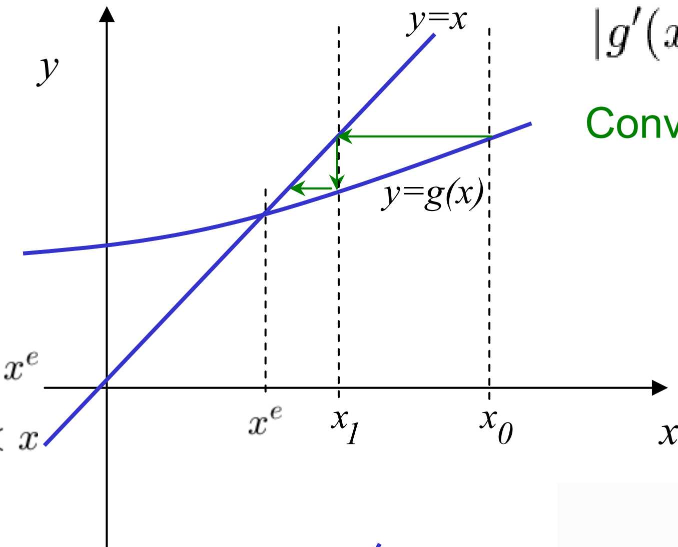
$$x_0 \in I, \quad k < 1$$

Roots of Nonlinear Equations
General Method

Convergence

$$|g'(x)| < 1$$

Convergent



Mean-value Theorem

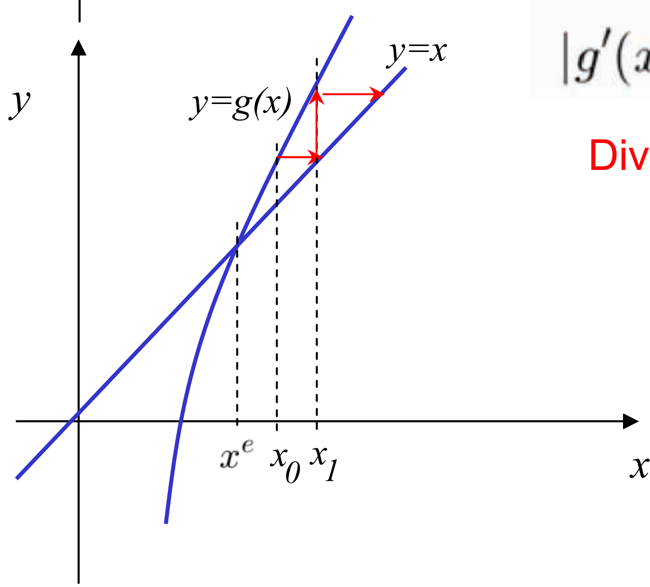
$$\{\exists \xi \in [x, x^e] \mid g(x) - g(x^e) = g'(\xi)(x - x^e)\} \begin{cases} x < \xi < x^e \\ x^e < \xi < x \end{cases}$$

Convergence

$$|g'(x)|_{x \in I} \leq k < 1 \Rightarrow |g(x) - x^e| \leq k|x - x^e|$$

$$|g'(x)| > 1$$

Divergent



Roots of Nonlinear Equations
General Method

Example: Cube root

$$x^3 - 2 = 0 \ , \quad x^e = 2^{1/3}$$

Rewrite

$$g(x) = x + C(x^3 - 2)$$

$$g'(x) = 3Cx^2 + 1$$

Convergence

$$|g'(x)| < 1 \quad \Leftrightarrow \quad -2 < 3Cx^2 < 0$$

$$\Leftrightarrow \quad \boxed{-1/6 < C < 0}$$

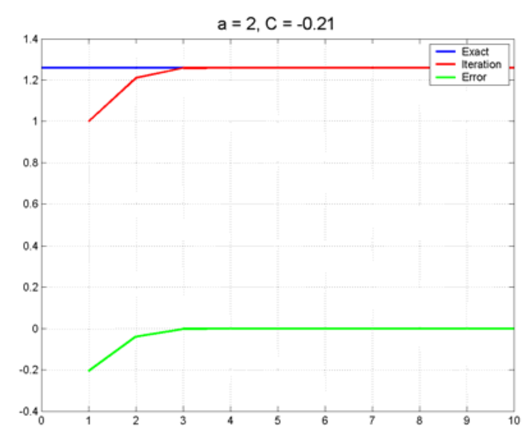
$$C = -\frac{1}{6} \Rightarrow x_{n+1} = g(x_n) = x_n - \frac{1}{6}(x_n^3 - 2)$$

Converges more rapidly for small $|g'(x)|$

$$g'(1.26) = 3C \cdot 1.26^2 + 1 = 0 \Leftrightarrow \boxed{C = -0.21}$$

cube.m

```
n=10;
g=1.0;
C=-0.21;
    sq(1)=g;
    for i=2:n
     sq(i)= sq(i-1) + C*(sq(i-1)^3 -a);
    end
    hold off
    f=plot([0 n],[a^(1./3.) a^(1/3.)],'b')
    set(f,'LineWidth',2);
    hold on
    f=plot(sq,'r')
    set(f,'LineWidth',2);
    f=plot( (sq-a^(1./3.))/(a^(1./3.)),'g')
    set(f,'LineWidth',2);
    legend('Exact','Iteration','Error');
    f=title(['a = ' num2str(a) ', C = ' num2str(C)])
    set(f,'FontSize',16);
    grid on
```



Roots of Nonlinear Equations
General Method

Converging, but how close?

$$
\begin{aligned}
|x_{n-1} - x^e| &\leq |x_{n-1} - x_n| + |x_n - x^e| \\
&= |x_{n-1} - x_n| + |g(x_{n-1}) - g(x^e)| \\
&= |x_{n-1} - x_n| + |g'(\xi)||x_{n-1} - x^e| \\
&\leq |x_{n-1} - x_n| + k|x_{n-1} - x^e| \\
&\Rightarrow \\
|x_{n-1} - x^e| &\leq \frac{1}{1-k}|x_{n-1} - x_n|
\end{aligned}
$$

Absolute error

$$
|x_n - x^e| \leq k|x_{n-1} - x^e| \leq \frac{k}{1-k}|x_{n-1} - x_n|
$$

General Convergence Rule

$$x_{n+1} = g(x_n)$$

$$|x_n - x^e| \leq \frac{k}{1-k}|x_{n-1} - x_n|$$

$$|g'(x)| < k < 1\,, \quad x \in I$$

Roots of Nonlinear Equations
Newton-Raphson Method

Non-linear Equation

$$f(x) = 0 \Leftrightarrow x = g(x)$$

Convergence Criteria

$$|g'(x_n)| < k < 1 \Rightarrow |x_n - x^e| \leq k|x_{n-1} - x^e|$$

Fast Convergence

$$|g'(x^e)| = 0$$

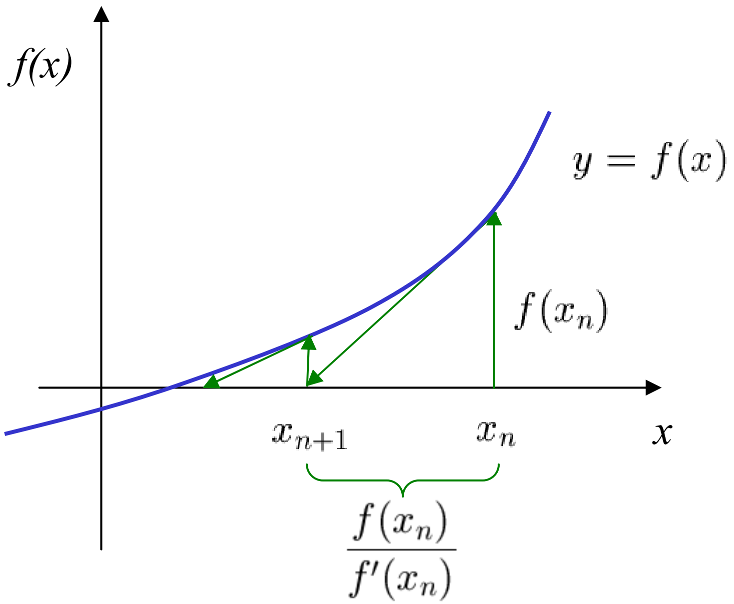
$$g(x) = x + h(x)f(x)\ , \quad h'(x) \neq 0$$

$$\begin{aligned} g'(x^e) &= 1 + h(x^e)f'(x^e) + h'(x^e)f(x^e) \\ &= 1 + h(x^e)f'(x^e) \end{aligned}$$

$$g'(x^e) = 0 \Leftarrow h(x) = -\frac{1}{f'(x)}$$

Newton-Raphson Iteration

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}$$



Roots of Nonlinear Equations
Newton-Raphson Method

Example – Square Root

$$x = \sqrt{a} \Leftrightarrow f(x) = x^2 - a = 0$$

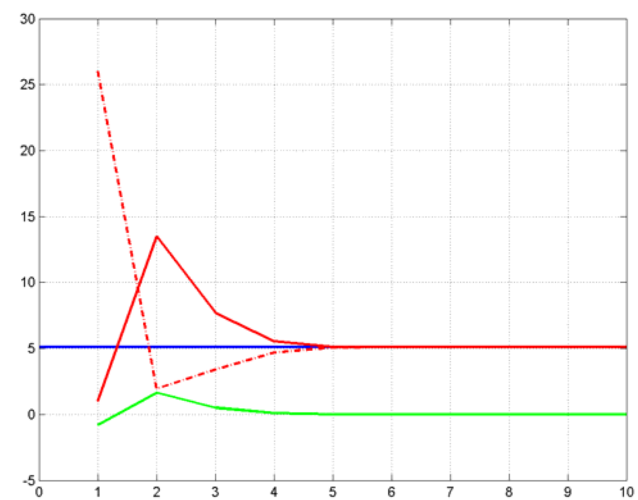
Newton-Raphson

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

Same as Heron's formula

sqr.m

```
a=26;
n=10;
g=1;
     sq(1)=g;
     for i=2:n
      sq(i)= 0.5*(sq(i-1) + a/sq(i-1));
     end
     hold off
     plot([0 n],[sqrt(a) sqrt(a)],'b')
     hold on
     plot(sq,'r')
     plot(a./sq,'r-.')
     plot((sq-sqrt(a))/sqrt(a),'g')
     grid on
```



Roots of Nonlinear Equations
Newton-Raphson Method

$$x = \frac{1}{a}$$

$$f(x) = ax - 1 = 0$$

$$f'(x) = a$$

Approximate Guess

$$\frac{|x - x^e|}{|x^e|} \ll 1$$

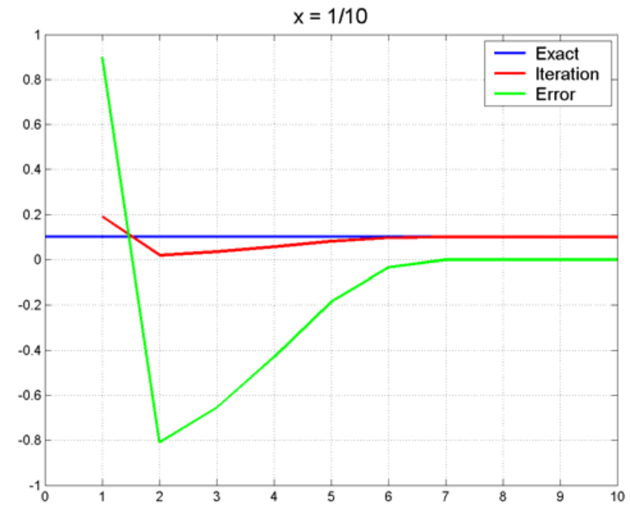
$$\frac{f(x)}{f'(x)} = \frac{ax - 1}{a} = x^e(ax - 1) \simeq x(ax - 1)$$

Newton-Raphson

$$x_{n+1} = x_n - x_n(ax_n - 1)$$

div.m

```
a=10;
n=10;
g=0.19;
sq(1)=g;
     for i=2:n
      sq(i)=sq(i-1) - sq(i-1)*(a*sq(i-1) -1) ;
     end
     hold off
     plot([0 n],[1/a 1/a],'b')
     hold on
     plot(sq,'r')
     plot((sq-1/a)*a,'g')
     grid on
     legend('Exact','Iteration','Error');
     title(['x = 1/' num2str(a)])
```



Roots of Nonlinear Equations
Newton-Raphson Method

Convergence Speed

$$\epsilon_n = x_n - x^e$$

Taylor Expansion

$$g(x_n) = g(x^e) + \epsilon_n g'(x^e) + \frac{1}{2}\epsilon_n^2 g''(x^e) \cdots$$

Second Order Expansion

$$g(x_n) - g(x^e) \simeq \frac{1}{2}\epsilon_n^2 g''(x^e)$$

$$\Rightarrow$$

$$\epsilon_{n+1} = x_{n+1} - x_e \simeq \frac{1}{2}\epsilon_n^2 g''(x^e)$$

Relative Error

$$\frac{\epsilon_{n+1}}{|x^e|} \simeq \frac{1}{2}|x^e| g''(x^e) \left(\frac{\epsilon_n}{|x^e|}\right)^2 = A(x^e)\left(\frac{\epsilon_n}{|x^e|}\right)^2$$

Quadratic Convergence

General Convergence Rate

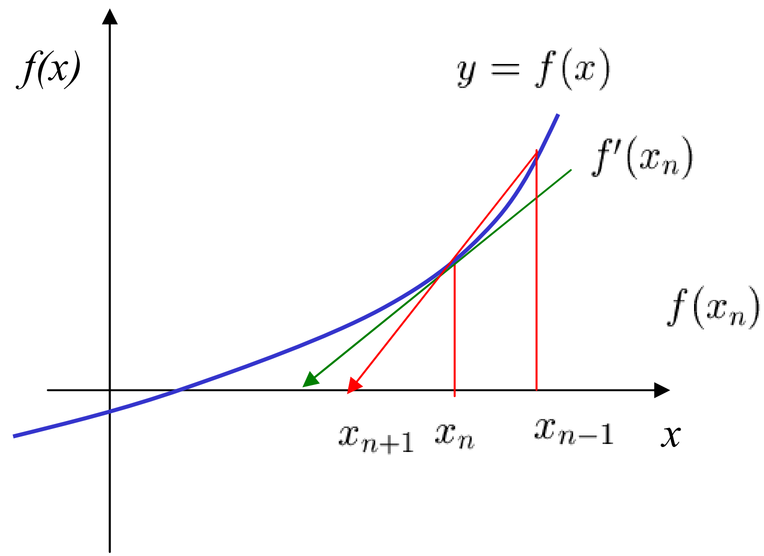
$$\epsilon_{n+1} \simeq \epsilon_n^m A$$

Convergence Exponent

Roots of Nonlinear Equations
Secant Method

1. In Newton-Raphson we have to evaluate 2 functions $f_n(x)$, $f'_n(x)$

2. $f_n(x)$ may not be given in closed, analytical form, i.e. it may be a result of a numerical algorithm



Approximate Derivative

$$f'(x_n) \simeq \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

Secant Method Iteration

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \\ &= \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})} \end{aligned}$$

Only 1 function call per iteration: $f_n(x)$

Roots of Nonlinear Equations
Secant Method

Convergence Speed

Absolute Error

$$\epsilon_n = x_n - x^e$$

$$\epsilon_{n+1} = x_{n+1} - x^e = \frac{f(x^e+\epsilon_n)(x^e+\epsilon_{n-1}) - f(x^e+\epsilon_{n-1})(x^e+\epsilon_n)}{f(x^e+\epsilon_n) - f(x^e+\epsilon_{n-1})} - x^e$$

Taylor Series – 2nd order

$$\epsilon_{n+1} \simeq \frac{1}{2}\epsilon_{n-1}\epsilon_n \frac{f''(x^e)}{f'(x^e)}$$

Relative Error

$$\frac{\epsilon_{n+1}}{|x^e|} \simeq \frac{\epsilon_{n-1}}{|x_e|}\frac{\epsilon_n}{|x_e|}\frac{f''(x^e)}{2f'(x^e)}x^e$$

Error Exponent

$$\epsilon_n = A(x^e)\epsilon_{n-1}^m \Rightarrow \epsilon_{n-1} = \left(\frac{1}{A}\epsilon_n\right)^{1/m} = B(x^e)\epsilon_n^{1/m}$$

$$\epsilon_{n+1} = C(x^e)\epsilon_n\epsilon_{n-1} = D(x^e)\epsilon_n\epsilon_n^{1/m} = D(x^e)\epsilon_n^{1+1/m}$$

$$1 + \frac{1}{m} = m \Leftrightarrow m = \frac{1}{2}(1+\sqrt{5}) \simeq 1.62$$

Error improvement for each function call

Secant Method $\quad \epsilon_{n+1}^* \simeq \epsilon_n^{1.62}$

Newton-Raphson $\quad \epsilon_{n+1}^* = \epsilon_n^{\sqrt{2}} \simeq \epsilon_n^{1.22}$

Exponents called Efficiency Index

Roots of Nonlinear Equations
Multiple Roots

p-order Root

$$f(x) = (x - x^e)^p f_1(x)\ ,\ f_1(x^e) \neq 0$$

Newton-Raphson

$$x_{n+1} = g(x_n) = x_n - \frac{(x_n - x^e)^p f_1(x_n)}{p(x_n - x^e)^{p-1} f_1(x_n) + (x_n - x^e)^p f'(x_n)}$$

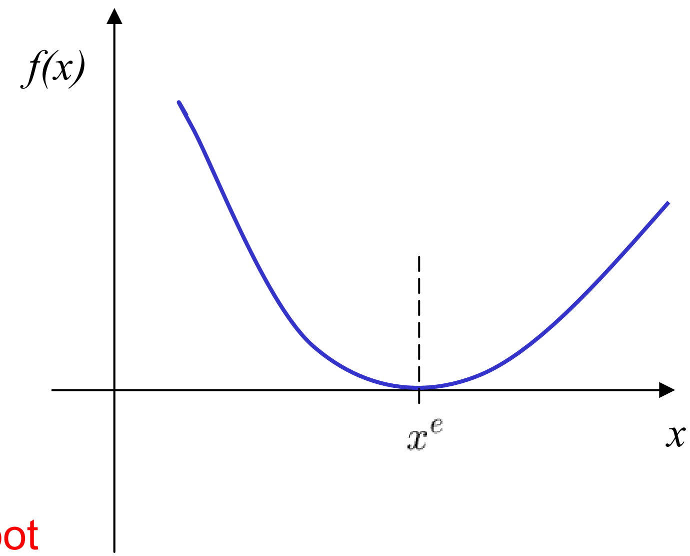
=>

$$x_{n+1} = x_n - \frac{(x_n - x^e) f_1(x_n)}{p f_1(x_n) + (x_n - x^e) f'(x_n)}$$

Convergence

$$|x_{n+1} - x^e| \leq k|x_n - x^e| \simeq |g'(x^e)|\,|x_n - x^e|$$

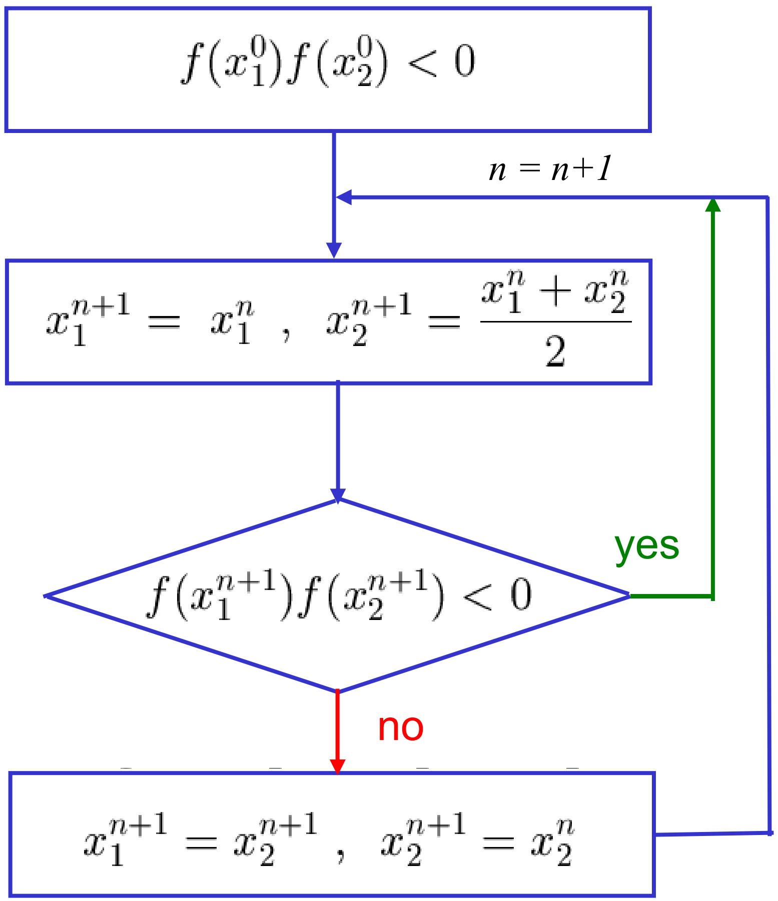
$$g'(x^e) = 1 - \frac{1}{p}$$

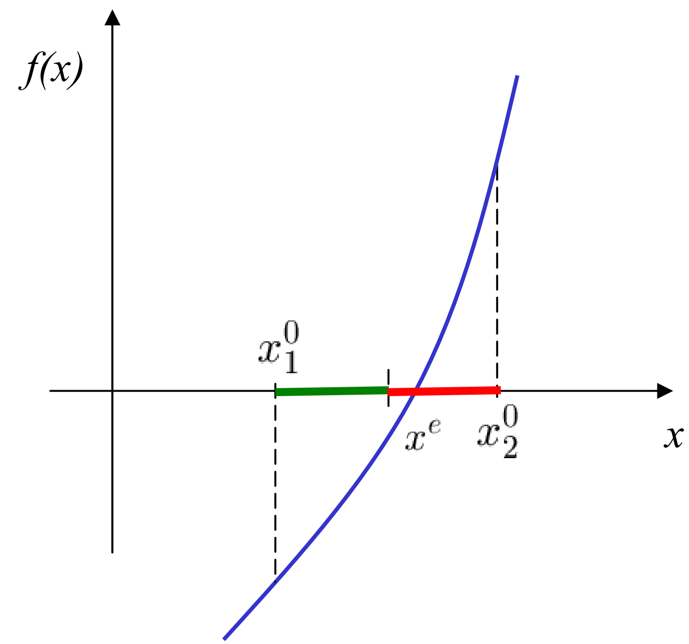


Slower convergence the higher the order of the root

Roots of Nonlinear Equations
Bisection

Algorithm

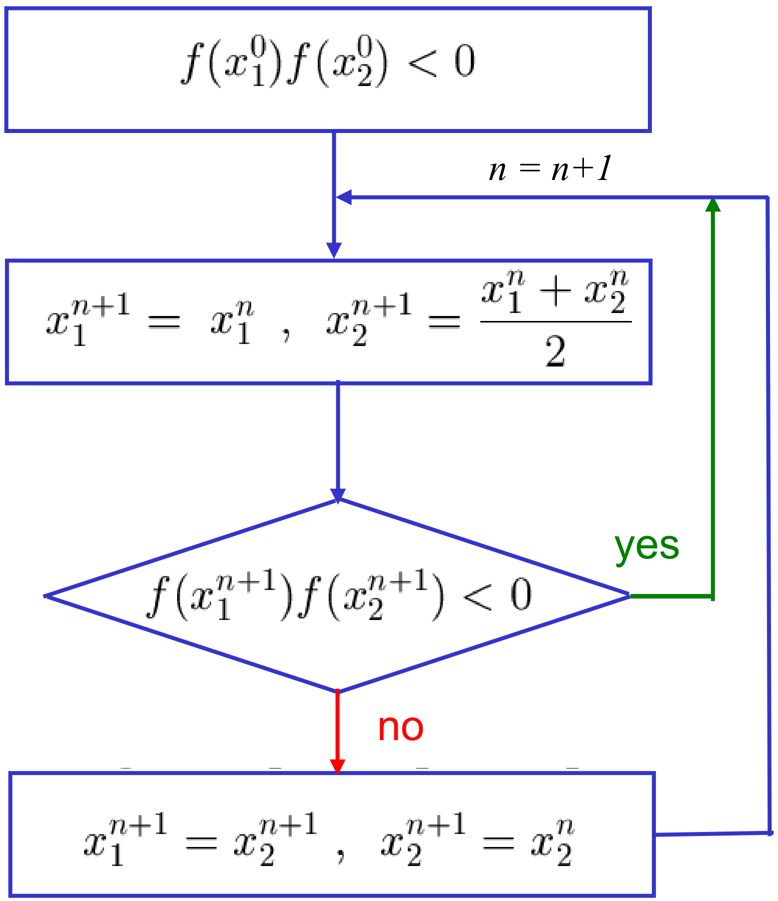




Less efficient than Newton-Raphson and Secant methods, but often used to isolate interval with root and obtain approximate value. Then followed by N-R or Secant method for accurate root.

Roots of Nonlinear Equations
Bisection

Algorithm



bisect.m

```
% Root finding by bi-section
f=inline(' a*x -1','x','a');
a=2
figure(1); clf; hold on
x=[0 1.5]; eps=1e-3;
err=max(abs(x(1)-x(2)),abs(f(x(1),a)-f(x(2),a)));
while (err>eps & f(x(1),a)*f(x(2),a) <= 0)
    xo=x; x=[xo(1) 0.5*(xo(1)+xo(2))];
    if ( f(x(1),a)*f(x(2),a) > 0 )
        x=[0.5*(xo(1)+xo(2)) xo(2)]
    end
    x
    err=max(abs(x(1)-x(2)),abs(f(x(1),a)-f(x(2),a)));
    b=plot(x,f(x,a),'.b'); set(b,'MarkerSize',20);
    grid on;
end
```