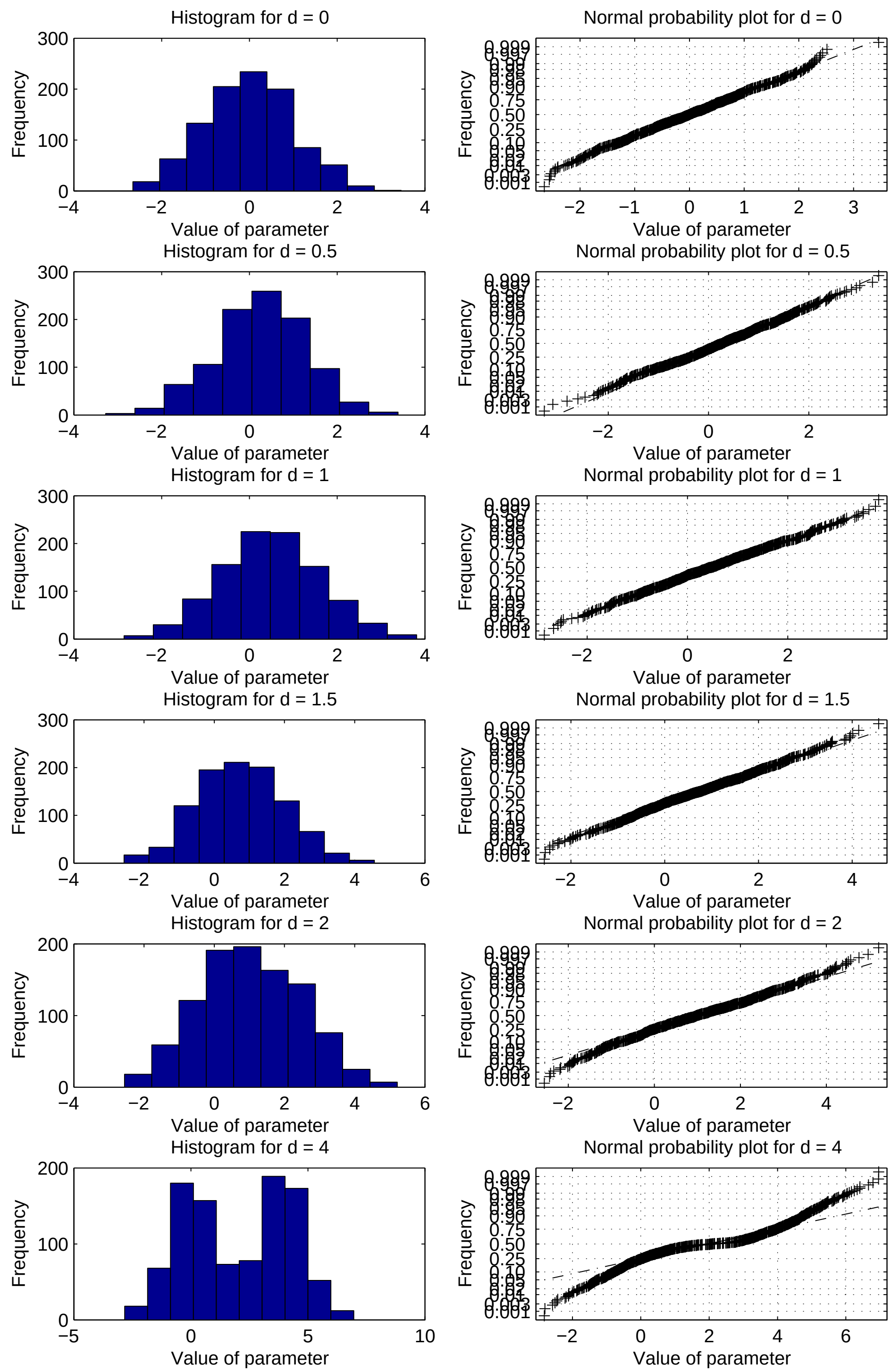
2.830J / 6.780J / ESD.63J Control of Manufacturing Processes (SMA 6303)
Spring 2008

MIT 2.830/6.780 Problem Set 3 (2008) — Solutions

Part 1

Histograms and normal probability plots for intermingled samples taken from two populations, $x_1 \sim N(0,1)$ and $x_2 \sim N(d,1)$, for values of d between 0 and 4:



Simply looking at these normal probability plots, one would probably conclude that the distributions underlying the samples for values of *d* up to and including 2 could be reasonably approximated by a normal distribution. Only for the case *d*=4 is the sample clearly from a non-normal distribution.

We might use various tests of normality to probe further. For this particular set of samples, the Lilliefors test rejects the hypothesis of normality at the 5% level for the cases *d*=2 and *d*=4. However, repeating the random sampling operation a few times shows that this is not always the result: depending on the samples that happen to be generated, the hypothesis of normality is sometimes rejected for *d* = 1 and *d* = 1.5.

So while normal probability plots and tests of normality are useful in deciding whether or not we can approximate a particular distribution as normal — in order, for example, to allow further hypothesis testing — they cannot be relied upon to alert us to features of the data that we had already inadvertently ignored.

Part 2

Montgomery problem 3-3

$\bar{x} = 26.0$

$s = 1.62$

$\mu_0 = 25$

$\alpha = 0.05$

$n = 10$

(a) Test H_0: $\mu = 25$ *vs* H_1: $\mu > 25$
Reject H_0 if $t_0 > t_\alpha$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{26.0 - 25}{1.62/\sqrt{10}} = 1.952$$

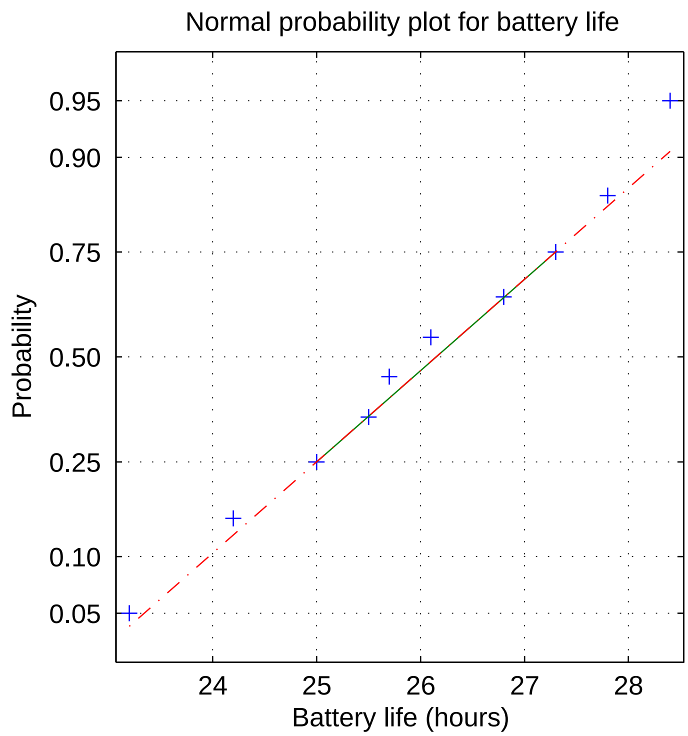
$t_{\alpha,\, n-1} = t_{0.05,\, 10-1} = 1.833$

Reject H_0, and conclude that the mean life exceeds 25 h.

(b) $\alpha = 0.10$

$$\bar{x} - t_{\alpha/2,n-1} s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1} s/\sqrt{n}$$
$$26.0 - 1.833(1.62/\sqrt{10}) \le \mu \le 26.0 + 1.833(1.62/\sqrt{10})$$
$$25.06 \le \mu \le 26.94$$

(c) Normal probability plot:



The plotted points fall approximately along a straight line, so the assumption that battery life is normally distributed seems appropriate.

Montgomery problem 3-6

$\bar{x} = 12.015$

$s = 0.030$

$\mu_0 = 12$

$\alpha = 0.01$

$n = 10$

(a) Test H_0: $\mu = 12$ *vs* H_1: $\mu > 12$
Reject H_0 if $t_0 > t_\alpha$

$$t_0 = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{12.015 - 12}{0.0303/\sqrt{10}} = 1.5655$$

$t_{\alpha,\, n-1} = t_{0.005,\, 10-1} = 3.520$

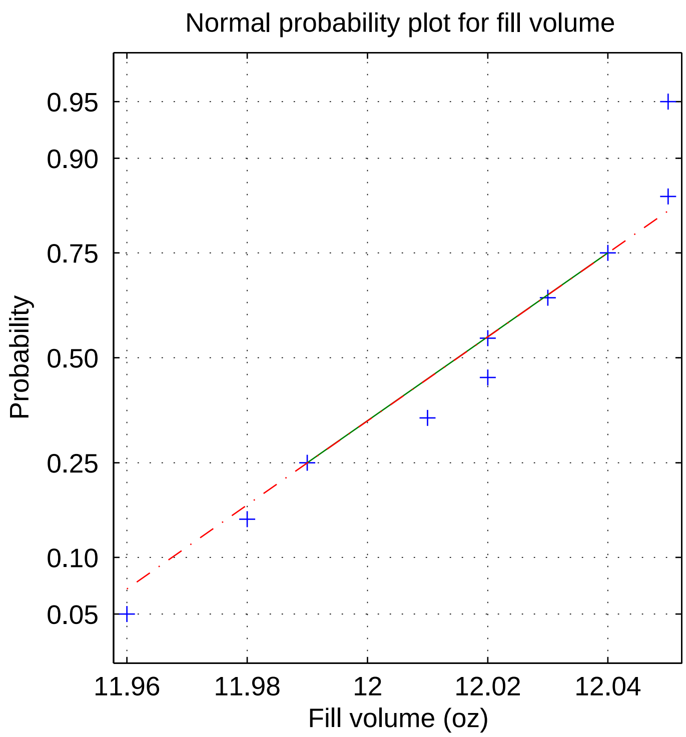
Do not reject H_0, and conclude that there is not enough evidence that the mean fill volume exceeds 12 oz.

(b) $\alpha = 0.05$

$t_{\alpha,\, n-1} = t_{0.025,\, 9} = 2.262$

$$\bar{x} - t_{\alpha/2,n-1} s/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2,n-1} s/\sqrt{n}$$
$$12.015 - 2.262(0.0303/\sqrt{10}) \le \mu \le 12.015 + 2.262(0.0303/\sqrt{10})$$
$$11.993 \le \mu \le 12.037$$

(c) Normal probability plot:



The plotted points fall approximately along a straight line, so the assumption that fill volume is normally distributed is appropriate. However the small sample size makes it difficult to be confident in this assumption.

Montgomery problem 3-11

(a) Let subscript 1 denote measurements by technician 1 and subscript 2 correspond to technician 2.

$\bar{x}_1 = 1.383$

$\bar{x}_2 = 1.376$

$s_1 = 0.115$

$s_2 = 0.125$

$n_1 = 7$

$n_2 = 8$

H_0: mean surface measurements made by the two technicians are equal.
H_1: mean surface measurements are different

We assume that the variances of populations 1 and 2 are equal, and calculate the pooled standard deviation for the samples as follows:

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} = 0.1204$$

The test statistic is

$$t_0 = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = 0.1061$$

At the 5% level, for a 2-tailed test, the critical value of the *t*-statistic is $t_{0.025,\, n_1+n_2-2} = 2.160$.

Since $t_0 < 2.160$, we do not reject H_0: there is insufficient evidence of a difference between mean measurements by the two technicians.

(b) The practical implication of this test is that the mean outcome of the measurement process is not dependent upon which of the two operators is carrying out the measurements. It does not, however, rule out the possibility of a substantial, consistent, and operator-independent error in any measurements taken. If the null hypothesis had been rejected, the validity of the measurements would have been cast into doubt. We would have needed to investigate the source of the difference — for example, whether the measurement procedure was not precisely enough defined, or whether one or both of the operators was not following the procedure properly.

(c) Using the values of s_p and critical t found above, the confidence interval is

$$(\bar{x}_1 - \bar{x}_2) - t_{\alpha/2, n_1+n_2-2}\, s_p\sqrt{1/n_1 + 1/n_2} \le \mu_1 - \mu_2 \le (\bar{x}_1 - \bar{x}_2) + t_{\alpha/2, n_1+n_2-2}\, s_p\sqrt{1/n_1 + 1/n_2}$$

$$(1.383 - 1.376) - 2.1604(0.12)\sqrt{(1/7) + (1/8)} \le \mu_1 - \mu_2 \le (1.383 - 1.376) + 2.1604(0.12)\sqrt{(1/7) + (1/8)}$$

$$-0.127 \le \mu_1 - \mu_2 \le 0.141$$

(d) $\alpha = 0.05$
Test H_0: $\sigma_1^2 = \sigma_2^2$ versus H_1: $\sigma_1^2 \neq \sigma_2^2$.

Reject H_0 if $F_0 > F_{\alpha/2,n_1-1,n_2-1}$ or $F_0 < F_{1-\alpha/2,n_1-1,n_2-1}$.

$F_0 = s_1^2/s_2^2 = 0.115^2/0.125^2 = 0.8464$.
$F_{\alpha/2,n_1-1,n_2-1} = F_{0.05/2,7-1,8-1} = 5.119$.
$F_{1-\alpha/2,n_1-1,n_2-1} = F_{0.975,7-1,8-1} = 0.176$.

Do not reject H_0 and conclude that there is insufficient evidence of a difference in variability of measurements obtained by the two technicians. Had the null hypothesis been rejected, we would have needed to investigate the source of any difference, such as one operator's being less careful than the other.

(e) $\alpha = 0.05$
$F_{\alpha/2,n_2-1,n_1-1} = 5.6955$; $F_{1-\alpha/2,n_2-1,n_1-1} = 0.1954$.

95% confidence interval estimate of the ratio of variances of technician measurements:

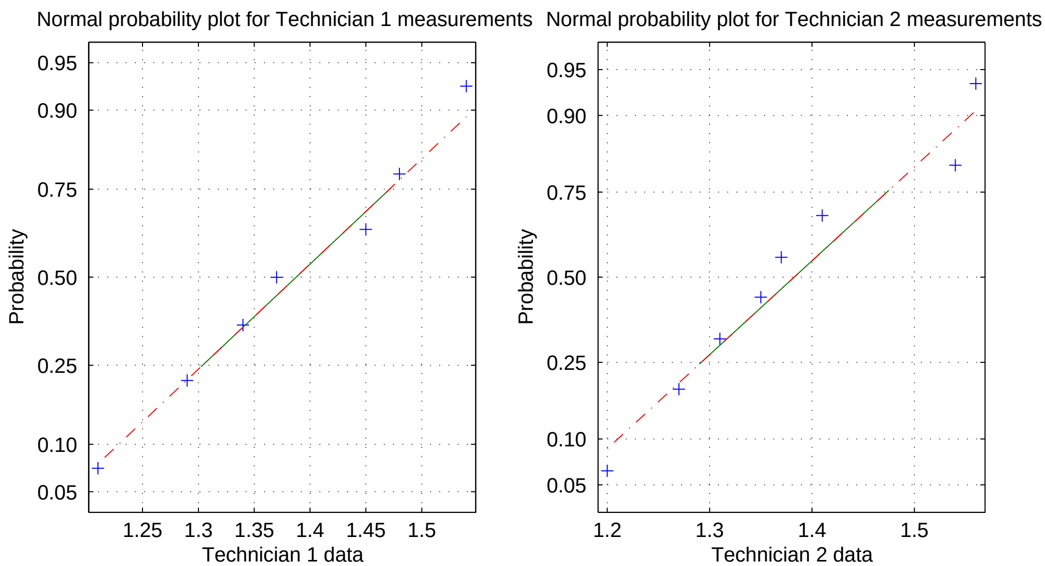
$$\frac{s_1^2}{s_2^2} F_{1-\alpha/2,n_2-1,n_1-1} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{s_1^2}{s_2^2} F_{\alpha/2,n_2-1,n_1-1}$$
$$\frac{0.115^2}{0.125^2}(0.1954) \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{0.115^2}{0.125^2}(5.6955)$$
$$0.165 \le \frac{\sigma_1^2}{\sigma_2^2} \le 4.821$$

(f) $\alpha = 0.05$
$\chi^2_{\alpha/2,n_2-1} = 16.0128$; $\chi^2_{1-\alpha/2,n_2-1} = 1.6899$

95% confidence interval estimate of the variance of measurements by technician 2:

$$\frac{(n_2-1)s_2^2}{\chi^2_{\alpha/2,n_2-1}} \le \sigma_2^2 \le \frac{(n_2-1)s_2^2}{\chi^2_{1-\alpha/2,n_2-1}}$$
$$\frac{(8-1)0.125^2}{16.0128} \le \sigma_2^2 \le \frac{(8-1)0.125^2}{1.6899}$$
$$0.007 \le \sigma_2^2 \le 0.065$$

(g)



The normality assumption seems reasonable for these readings.

Montgomery problem 3-21

$\bar{x} = 752.6\ mL$

$n = 20$

$s = 1.5\ mL$

$\alpha = 0.05$

(a) Test H_0: $\sigma^2 = 1$ versus H_1: $\sigma^2 < 1$. Reject H_0 if $\chi^2_0 < \chi^2_{1-\alpha,n-1}$.

$\chi^2_{1-\alpha,n-1} = \chi^2_{0.95,19} = 10.1170$.

$$\chi_0^2 = \frac{(n-1)s^2}{\sigma_0^2} = \frac{(20-1)1.5^2}{1} = 42.75$$

$\chi^2_0 = 42.75 > 10.12$, so do not reject H_0. There is no significant evidence that the standard deviation of the fill volume is less than 1 mL.

(b) 95% two-sided confidence interval on the standard deviation of fill volume:

$$\chi^2_{\alpha/2,n-1} = \chi^2_{0.025,19} = 32.85$$

$$\chi^2_{1-\alpha/2,n-1} = \chi^2_{0.975,19} = 8.91$$

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

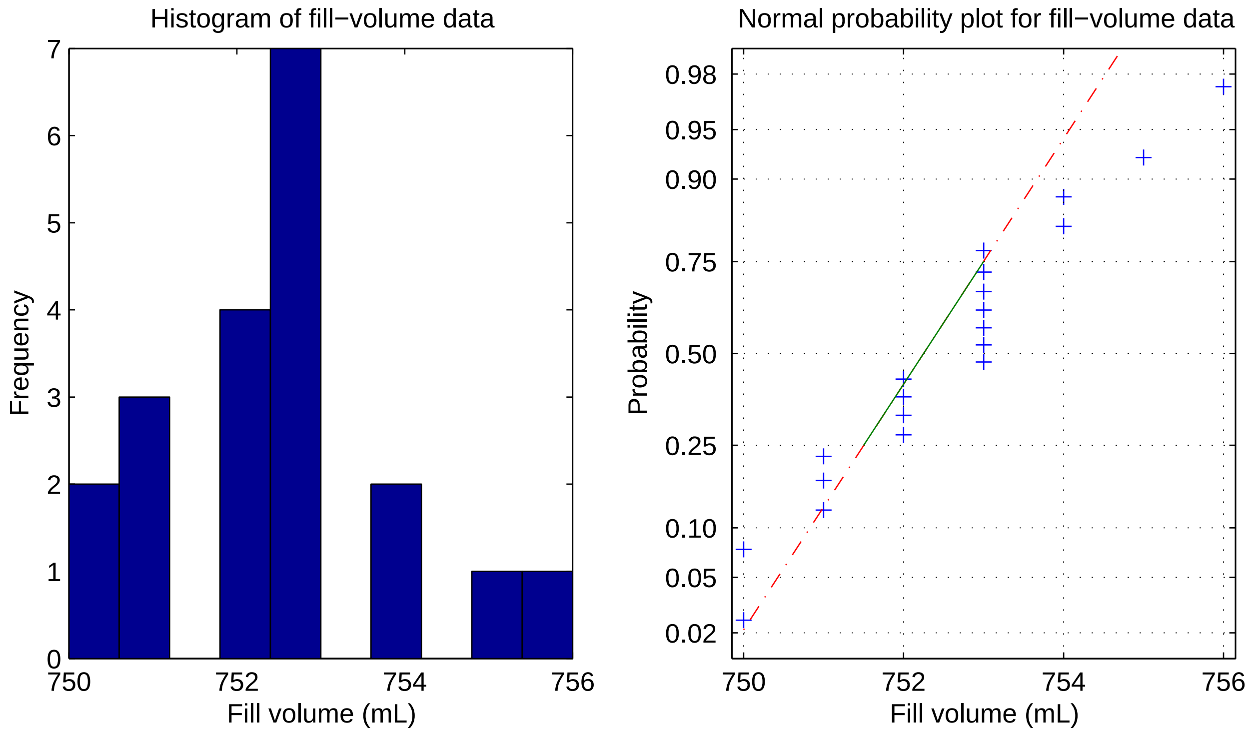
$$\frac{(20-1)1.5^2}{32.85} \le \sigma^2 \le \frac{(20-1)1.5^2}{8.91}$$

$$1.30 \le \sigma^2 \le 4.80$$

$$1.14 \le \sigma \le 2.19$$

Units of σ: mL.

(c) Histogram and normal probability plot for fill-volume data



The kurtosis is 2.86 (normal: 3), and the skewness 0.26, suggesting that the data are reasonably represented by a normal distribution. However, the normal probability plot shows the data deviating substantially from a straight line, as well as highlighting the substantial quantization of the data – whether this is real or from measurement we cannot say. An assumption of normality is therefore rather dubious.

Part 3

May and Spanos, *problem 6.1*

What is the probability of 4 out of 5 consecutive points plotting outside the $\pm 1\sigma$ limits?

[The question said '2σ' limits; answers interpreting this as $\pm 2\sigma$ will be dealt with leniently.]

We interpret the question as asking for the probability of obtaining this warning when the process is in control. Note that the four points have to be on the *same* side of the center-line to cause a warning to be triggered.

So the probability is:

$(2 \text{ sides}) \times {}^5C_4 \times [1-\Phi(1)]^4[\Phi(1)] = 2 \times 5 \times (1-0.8413)^4(0.8413) = 5.34 \times 10^{-3}$.

We also assume that the triggering of a warning does not cause the process to be suspended; if it did, the above calculation would no longer hold.

I suppose the rule should strictly be '*at least*' 4 out of the last 5 points, since a continuous run of 5 points on one side of the 1σ limits would be at least as suggestive of an out-of-control process as four out of five points. We might therefore justify adding
$2 \times [1-\Phi(1)]^5$ to the probability of a warning being triggered.

May and Spanos, *problem 6.2*

The probability of observing a type I error (reject the hypothesis that the process is in control when it *is* in control) is

$2\,[1-\Phi(3)] = 0.0027$.

May and Spanos, problem 6.3

(a) P(signal alarm on second sample after the shift) = $\beta(1-\beta)$

(b) P(miss the alarm for K samples following the shift) = β^K

If this statement is interpreted as meaning 'at least K samples' the required probability is β^K; if it is interpreted as meaning 'exactly K samples', the probability is $\beta^K(1-\beta)$: the alarm would be triggered on sample K+1.

(c) The expected number of samples needed after the shift to generate an alarm is $1/(1-\beta)$ (the out-of-control average run length).

Part 4

Montgomery problem 7-3

$$\hat{\mu} = \overline{\overline{x}} = 10.375;\ \overline{R}_x = 6.25;\ \hat{\sigma}_x = \overline{R}/d_2 = 6.25/2.059 = 3.04$$

$$\text{USL}_x = [(350+5)-350]\times 10 = 50;\ \text{LSL}_x = [(350-5)-350]\times 10 = -50$$

$$x_i = (\text{obs}_i - 350)\times 10$$

$$\hat{C}_p = \frac{\text{USL}_x - \text{LSL}_x}{6\hat{\sigma}_x} = \frac{50-(-50)}{6(3.04)} = 5.48$$

The process produces product that uses approximately 18% of the total specification band.

$$\hat{C}_{pu} = \frac{\text{USL}_x - \hat{\mu}}{3\hat{\sigma}_x} = \frac{50-10.375}{3(3.04)} = 4.34$$

$$\hat{C}_{pl} = \frac{\hat{\mu} - \text{LSL}_x}{3\hat{\sigma}_x} = \frac{10.375-(-50)}{3(3.04)} = 6.62$$

$$\hat{C}_{pk} = \min(\hat{C}_{pu}, \hat{C}_{pl}) = 4.34$$

This is an extremely capable process, with an estimated percent defective much less than 1 ppb. Note that the C_{pk} is less than C_p, indicating that the process is not centered and is not achieving potential capability. However, this PCR does not tell *where* the mean is located within the specification band.

$$V = \frac{T - \overline{\overline{x}}}{S} = \frac{0-10.375}{3.04} = -3.4128$$

$$\hat{C}_{pm} = \frac{\hat{C}_p}{\sqrt{1+V^2}} = \frac{5.48}{\sqrt{1+(-3.4128)^2}} = 1.54$$

Since C_{pm} is greater than 4/3, the mean μ lies within approximately the middle fourth of the specification band.

$$\hat{\xi} = \frac{\hat{\mu} - T}{\hat{\sigma}} = \frac{10.375 - 0}{3.04} = 3.41$$

$$\hat{C}_{pkm} = \frac{\hat{C}_{pk}}{\sqrt{1 + \hat{\xi}^2}} = \frac{4.34}{\sqrt{1 + 3.4128^2}} = 1.22$$

Montgomery problem 7-6

[numbers in the 4th Edition are different; solutions using them were accepted]

$n = 4; \quad \hat{\mu} = \overline{\overline{x}} = 199; \quad \overline{R} = 3.5; \quad \hat{\sigma}_x = \overline{R}/d_2 = 3.5/2.059 = 1.70$

USL = 200 + 8 = 208; LSL = 200 – 8 = 192

(a)

Potential: $\hat{C}_p = \dfrac{\text{USL} - \text{LSL}}{6\hat{\sigma}} = \dfrac{208 - 192}{6(1.70)} = 1.57$

The process produces product that uses approximately 64% of the total specification band.

(b)

$$\hat{C}_{pu} = \frac{\text{USL} - \hat{\mu}}{3\hat{\sigma}} = \frac{208 - 199}{3(1.70)} = 1.76$$

Actual: $\hat{C}_{pl} = \dfrac{\hat{\mu} - \text{LSL}}{3\hat{\sigma}} = \dfrac{199 - 192}{3(1.70)} = 1.37$

$$\hat{C}_{pk} = \min(\hat{C}_{pl}, \hat{C}_{pu}) = 1.37$$

(c)

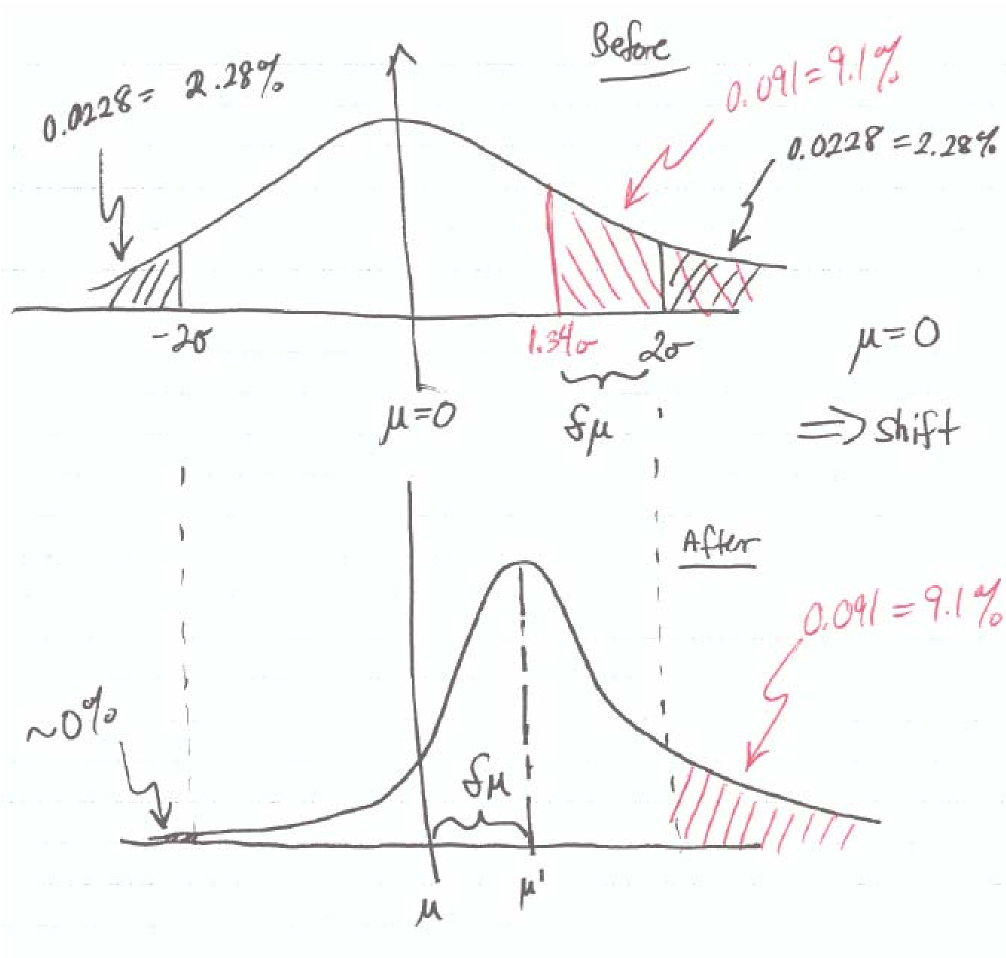
The current fraction nonconforming is:

$$\begin{aligned}
\hat{p}_{\text{Actual}} &= \Pr\{x < \text{LSL}\} + \Pr\{x > \text{USL}\} \\
&= \Pr\{x < \text{LSL}\} + \left[1 - \Pr\{x \le \text{USL}\}\right] \\
&= \Pr\left\{z < \frac{\text{LSL} - \hat{\mu}}{\hat{\sigma}}\right\} + \left[1 - \Pr\left\{z \le \frac{\text{USL} - \hat{\mu}}{\hat{\sigma}}\right\}\right] \\
&= \Pr\left\{z < \frac{192 - 199}{1.70}\right\} + \left[1 - \Pr\left\{z \le \frac{208 - 199}{1.70}\right\}\right] \\
&= \Phi(-4.1176) + \left[1 - \Phi(5.2941)\right] \\
&= 0.0000191 + \left[1 - 1\right] \\
&= 0.0000191
\end{aligned}$$

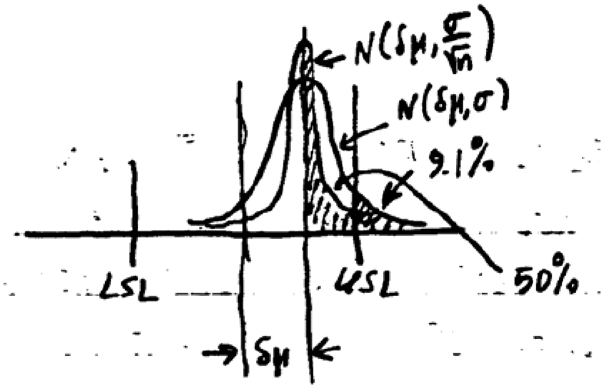
If the process mean could be centered at the specification target, the fraction nonconforming would be:

$$\begin{aligned}
\hat{p}_{\text{Potential}} &= 2 \times \Pr\left\{z < \frac{192 - 200}{1.70}\right\} \\
&= 2 \times 0.0000013 \\
&= 0.0000026
\end{aligned}$$

May and Spanos problem 6.8 - Solution partially removed due to copyright restrictions.



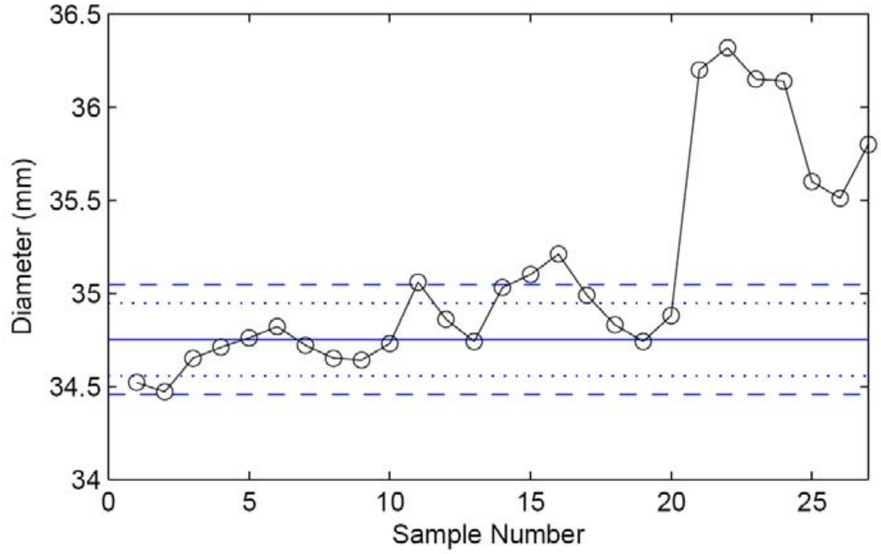
$$\delta\mu = \mu' - \mu = 0.66\sigma$$



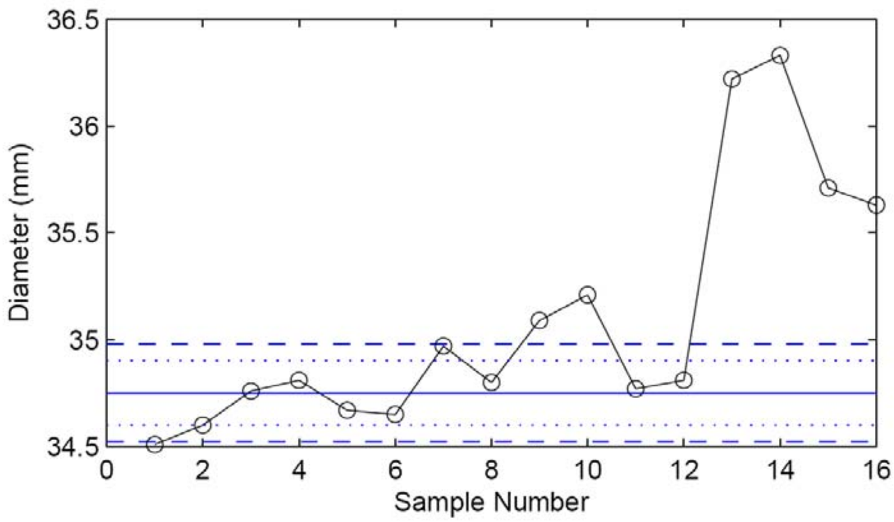
Part 5

(a) Control charts:

For sample size 3:

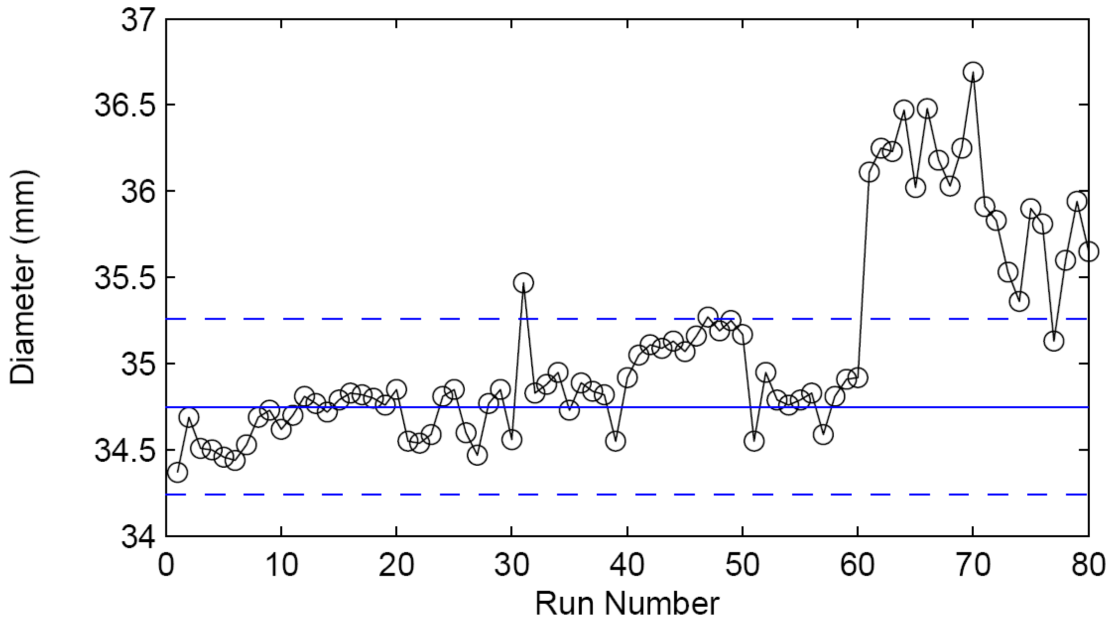


For sample size 5:



(b) With a sample size of 3, there is a warning from rule 2 (2 out of 3 consecutive points outside $\pm 2\sigma$) at samples 2 and 3, and alarms from points outside $\pm 3\sigma$ at samples 11, 16, 17 and 21–27. With a sample size of 5, there are alarms from points outside $\pm 3\sigma$ at samples 1, 9, 10 and 13–16.

(c) Run chart for $C_p = C_{pk} = 1$:



If applying the WECO rules strictly, with a sample size of 3, the process would be stopped at sample 2 to investigate the cause of the consecutive pair of samples below -2σ. Upon finding that the parts produced were not defective, the process would probably be restarted, and would continue until sample 11, when an out-of-control alarm is sounded. The out-of-specification part from run 31 would already have been produced. However, this out-of-spec part would probably be concluded to have been an 'outlier' and the process restarted. The next alarm is at sample 21, reacting to the 3 out-of-spec parts from runs 61–63. A clear mean shift would be discovered upon investigation, and the apparatus corrected. The defective run 47 would not have been detected. Thus, 5 out-of-spec parts are produced if a control chart with a sample size of 3 is used.

With a sample size of 5, the first alarm following an out-of-spec part is at sample 9, after 45 parts have been produced and only one defective part (the outlier at run 31) has been produced. The operator might look at the run chart, conclude that the process had been drifting during runs 40–45, and take corrective action. If they did *not* take corrective action, they would get another alarm at sample 10, by which time the defective run 47 would have been produced. Most operators would probably accept by this stage that the process was drifting, and take corrective action. The most cavalier operators would, however, restart the process without correction, and not get another alarm until sample 13, after defective runs 61–65 had occurred. So even if the WECO rules are applied rigorously, the judgement of the operator after the process has been stopped determines whether 1, 2 or 7 defective parts are produced.

(d) Average run length for detecting a mean shift of 0.25σ with sample size of 5, for example:

$$\beta = \Phi(L - k\sqrt{n}) - \Phi(-L - k\sqrt{n}) = \Phi(3 - 0.25\sqrt{5}) - \Phi(-3 - 0.25\sqrt{5}) = 0.9925$$

$$ARL = \frac{1}{1-\beta} = 133$$

(e) If no SPC had been performed (i.e. the run of 80 parts had been produced without any measurements during production), 21 defective parts would be produced: runs 31, 47, 61–76 and 78–80.