MASSACHUSETTS INSTITUTE OF TECHNOLOGY

2.71/2.710 Optics **Spring '14**

Practice Problems for the Final Exam Posted Monday, May 17, 2014

1. **(Pedrotti 13-21)** A glass plate is sprayed with uniform opaque particles. When a distant point source of light is observed looking through the plate, a diffuse halo is seen whose angular width is about 2^o. Estimate the size of the particles. (Hint: consider Fraunhoffer diffraction through random gratings, and use Babinet's principle)

Answer:

The diffraction pattern of an opaque circular particle is complementary to that due to circular apertures of the same size in an otherwise opaque screen.

Under the Fraunhofer condition ($\frac{k\left(x'^2+y'^2\right)}{2z} \ll 1,\ \frac{k\left(x^2+y^2\right)}{2z} \ll 1$)

$$E(x',y') \approx \frac{1}{z}\iint \exp(-ik(\theta_{x'}x+\theta_{y'}y))t(x,y)E(x,y)dxdy$$

Where $\theta_{x'} \approx \frac{x\prime}{z}, \theta_{y'} \approx \frac{y\prime}{z}$

For the given problem, we may further assume E(x, y) is a plane wave at normal incidence, and the transmission function t(x, y) for a single can be expressed as:

$$t(x,y) = 1 - circ(\frac{\sqrt{x^2+y^2}}{R})$$

Where R is the radius of the opaque particles.

$$E(x',y') \approx \frac{1}{z}\iint \exp(-ik(\theta_{x'}x+\theta_{y'}y))\left[1 - circ(\frac{\sqrt{x^2+y^2}}{R})\right]dxdy$$

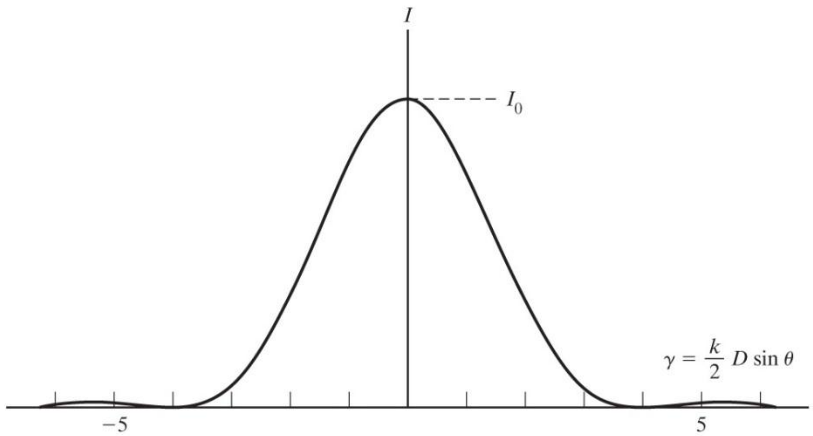
$$E(x',y') \approx \frac{1}{z}\mathcal{F}\left[1 - circ\left(\frac{\sqrt{x^2+y^2}}{R}\right)\right]$$

With $x' = \frac{z}{k}k_x\,, y' = \frac{z}{k}k_y$

$$E\left(k_x,k_y\right) \approx \frac{1}{z}\left[\delta(\sqrt{{k_x}^2+{k_y}^2}) - |R|^2\frac{2\pi J_1\left(R\sqrt{{k_x}^2+{k_y}^2}\right)}{R\sqrt{{k_x}^2+{k_y}^2}}\right]$$

The halo is similar to an Airy disc!

We can evaluate the width of the halo (a second peak) based on the table on Figure 11_08 provided by Pedrotti:



(a)

	γ	$I/I_0 = (2J_1(\gamma)/\gamma)^2$
1st Maximum	0	1
1st Zero	3.832	0
2nd Maximum	5.136	0.0175
2nd Zero	7.016	0
3rd Maximum	8.417	0.00416
3rd Zero	10.173	0
4th Maximum	11.620	0.00160
4th Zero	13.324	0

(b)

$$\text{Where } \gamma = R\sqrt{{k_x}^2 + {k_y}^2} = \frac{2\pi}{\lambda} R\theta.$$

From the above table,

$$\frac{2\pi}{\lambda} R\Delta\theta = 7.106 - 3.832 = 3.274$$

Taking central wavelength at visible frequency, $\lambda = 500$ nm and given $\Delta\theta = 2^o$, we find the radius of the particle:

$$R = \lambda \frac{3.274}{(2\pi)^2(\frac{\Delta\theta}{360})} = 500nm \times \left(\frac{3.274}{(2\pi)^2 \frac{2}{360}}\right) = 7463nm = 7.4\mu m$$

2. **(Adapted from Pedrotti 16-1 and 16-12)**

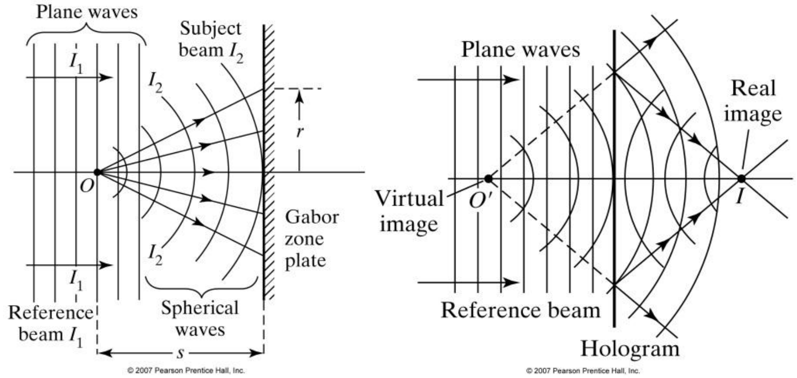


Figure A. Recording(Left) and Reconstruction(Right) of a Gabor Hologram

a) Use the superposition of two beams to show that the recorded intensity pattern on a Gabor zone-plate (the hologram of a point source) is given approximately by

$$I = A + Bcos^2(ar^2)$$

Where $A = I_1 + I_2 - 2\sqrt{I_1 I_2}$, $B = 4\sqrt{I_1 I_2}$, and $a = \pi/(2s\lambda)$. Here I_1 and I_2 are the intensity due to the reference and signal beams, respectively, s is the distance of the object point from the film, and λ is the wavelength of the light. For the approximation, assume the path difference between the two beams is much smaller than s, so we are looking at the inner zones of the hologram.

Solution: in this problem, two beams are interfering at the zone plate: a reference plane wave with intensity I_1, and a spherical wave with intensity I_2. At a distance r from the symmetric axis, the path difference of the two beams can be written as:

$$\delta = \sqrt{s^2 + r^2} - s \approx \frac{r^2}{2s}$$

Therefore the intensity of the interference pattern can be written as:

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2}cos(k\delta)$$

And

$$cos(k\delta) = 2cos^2\left(\frac{k\delta}{2}\right) - 1$$

So we can rewrite the intensity into the following form:

$$I = \left[I_1 + I_2 - 2\sqrt{I_1 I_2}\right] + 4\sqrt{I_1 I_2}cos^2\left(\frac{k\delta}{2}\right)$$

$$\frac{k\delta}{2} = \frac{\pi r^2}{2\lambda s}$$

b) (**2.710 only**) Show that the phase delay of the diverging subject beam, at a point on the film at distance r from the axis, is given by $\pi r^2/ls$. This results follows when r<<s. Show also that the amplitude of the light transmitted by the film under illumination of the reference beam produces converging spherical wavefront, thus a real image on reconstruction.

Answer:

The path difference δ of the diverging beam with respect to the plane wave is derived in part (a). Therefore the phase delay is:

$$k\delta = \frac{\pi r^2}{\lambda s}$$

To answer last part of the question we can calculate the Fresnel diffraction pattern of this system using $k_x = k\frac{x'}{z}$, $k_y = k\frac{y'}{z}$.

$$E(x', y') \approx \iint \exp(\,ik\frac{x^2+y^2}{2z})\left\{1 + \cos\left[k\,\frac{x^2+y^2}{2s}\right]\right\}\exp\{-i[k_x x + k_y y]\}dxdy$$

$$E(x', y') \approx \mathcal{F}\left\{\exp(\,ik\frac{x^2+y^2}{2z}) + \frac{1}{2}\exp\left[ik(x^2+y^2)\left(\frac{1}{2s} + \frac{1}{2z}\right)\right] + \frac{1}{2}\exp\left[-ik(x^2+y^2)\left(\frac{1}{2s} - \frac{1}{2z}\right)\right]\right\}$$

The Fourier transform of the first term is straight forward:

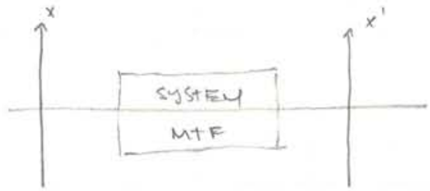
$$\exp\left(\,-ik\frac{x'^2+y'^2}{2z}\right).$$

Likewise, we can express the second and the third term:

$$\frac{1}{2}\exp\left(\,-ik\frac{s}{2z}\frac{x'^2+y'^2}{s+z}\right) + \frac{1}{2}\exp\left(\,ik\frac{s}{2z}\frac{x'^2+y'^2}{(z-s)}\right).$$

the 3rd term indicates a converging wave front towards z=s (**a real image**) on the optical axis.

(2)



Here we have some unknown optical system. But we know how it behaves under incoherent illumination because the MTF is provided.

a)

input intensity: $I(x) = \frac{1}{2}\left[1 + \frac{1}{2}\cos\left(\frac{2\pi x}{50\mu m}\right) + \frac{1}{2}\cos\left(\frac{2\pi x}{10\mu m}\right)\right]$

contrast is defined as: $\mathcal{V} = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}}$

from inspection, I_{max} @ $x=0 \rightarrow I_{max} = \frac{1}{2}\left[1 + \frac{1}{2}(1) + \frac{1}{2}(1)\right] = 1$

I_{min} @ $x = 25\mu m \rightarrow I_{min} = \frac{1}{2}\left[1 - \frac{1}{2} - \frac{1}{2}\right] = 0$

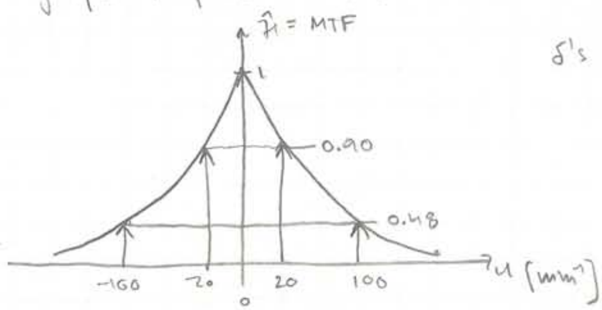
$\therefore \mathcal{V} = \dfrac{1-0}{1+0} = \boxed{1 = \mathcal{V}}$ perfect contrast

b) output is found by taking the FT of the input intensity, multiplying by the optical transfer function, $\mathcal{H}$ (MTF is magnitude of OTF) which gives the Fourier transform of the output intensity. Find the output intensity by taking the Fourier inverse transform

$$G^I_{in} = \mathcal{F}\{I_{in}\} = \mathcal{F}\left\{\frac{1}{2}\left[1 + \frac{1}{4}e^{i2\pi x/50\mu m} + \frac{1}{4}e^{-i2\pi x/50\mu m} + \frac{1}{4}e^{i2\pi x/10\mu m} + \frac{1}{4}e^{-i2\pi x/10\mu m}\right]\right\}$$

$$= \frac{1}{2}\left[\delta(u) + \frac{1}{4}\delta(u - 1/50\mu m) + \frac{1}{4}\delta(u + 1/50\mu m) + \frac{1}{4}\delta(u - 1/10\mu m) + \frac{1}{4}\delta(u + 1/10\mu m)\right]$$

now we must multiply G^I_{in} by the OTF. Since G^I_{in} consists only of delta functions and we are given the MTF = |OTF| we can simply use the provided graph to pick out the values at each δ



δ's located @ $u = \left\{0, \frac{1}{50\mu m}, -\frac{1}{50\mu m}, \frac{1}{10\mu m}, -\frac{1}{10\mu m}\right\}$

$\frac{1}{50\mu m}\cdot\frac{1000\mu m}{1mm} = \underline{20\ mm^{-1}}$

$\frac{1}{10\mu m}\cdot\frac{1000\mu m}{1mm} = \underline{100\ mm^{-1}}$

$$G^I_{out} = G^I_{in}\cdot\hat{\mathcal{H}} = \hat{\mathcal{H}}(0)\frac{\delta(u)}{2} + \frac{1}{8}\hat{\mathcal{H}}(20mm^{-1})\left[\delta(u - 1/50\mu m) + \delta(u + 1/50\mu m)\right] + \frac{1}{8}\hat{\mathcal{H}}(100m^{-1})\left[\delta(u - 1/10\mu m) + \delta(u + 1/10\mu m)\right]$$

$$G^I_{out} = (1)\frac{\delta(u)}{2} + \frac{1}{8}(0.9)\left[\delta(u - 1/50\mu m) + \delta(u + 1/50\mu m)\right] + \frac{1}{8}(0.48)\left[\delta(u - 1/10\mu m) + \delta(u + 1/10\mu m)\right]$$

$$G^I_{out} = \frac{1}{2}\left[\delta(u) + \frac{9}{40}\left(\delta(u - 1/50\mu m) + \delta(u + 1/50\mu m)\right) + \frac{3}{25}\left[\delta(u - 1/10\mu m) + \delta(u + 1/10\mu m)\right]\right]$$

now $I_{out} = \mathcal{F}^{-1}\{G^I_{out}\} = \frac{1}{2} + \frac{9}{80}\left[e^{i2\pi x/50\mu m} + e^{-i2\pi x/50\mu m}\right] + \frac{3}{50}\left[e^{i2\pi x/10\mu m} + e^{-i2\pi x/10\mu m}\right]$

$$\boxed{I_{out}(x') = \frac{1}{2}\left[1 + \frac{9}{20}\cos(2\pi x/50\mu m) + \frac{6}{25}\cos(2\pi x/10\mu m)\right]}$$

contrast: $\mathcal{V} = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}}$

I_{max} @ $x = 0 \rightarrow I_{max} = \left[1 + \frac{9}{20} + \frac{6}{25}\right] \frac{1}{2}$

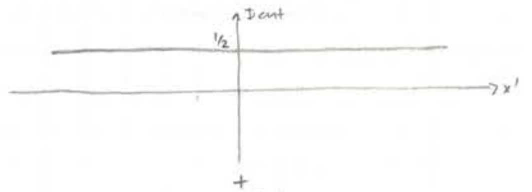
$I_{max} = 169/200$

I_{min} @ $x = 25\mu m \rightarrow I_{min} = \frac{1}{2}\left[1 - 9/20 - 6/25\right]$

$I_{min} = 31/200$

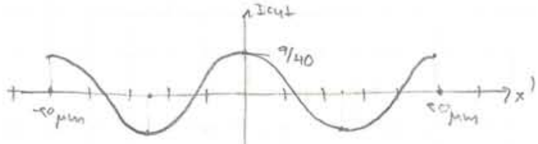
$$\mathcal{V} = \frac{\frac{169}{200} - \frac{31}{200}}{\frac{169}{200} + \frac{31}{200}}$$

$$= \frac{138}{200} = \boxed{\frac{69}{100} = \mathcal{V}}$$



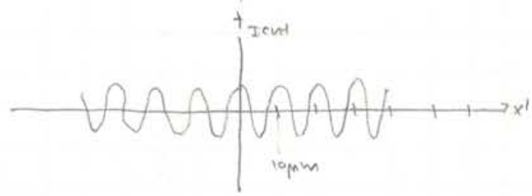
plot 1/2

+



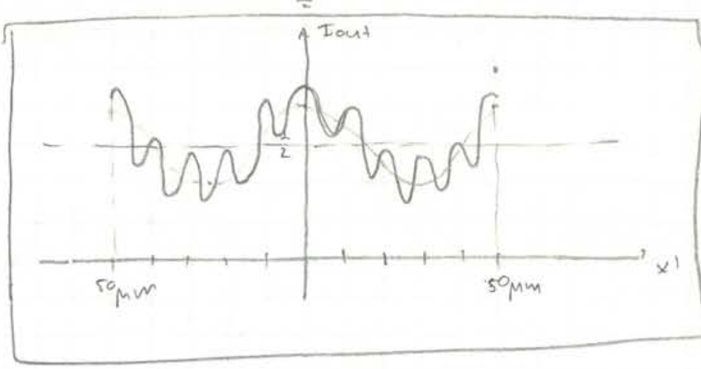
plot $\frac{9}{40}\cos(2\pi x/50\mu m)$

+



plot $\frac{6}{50}\cos(2\pi x/10\mu m)$

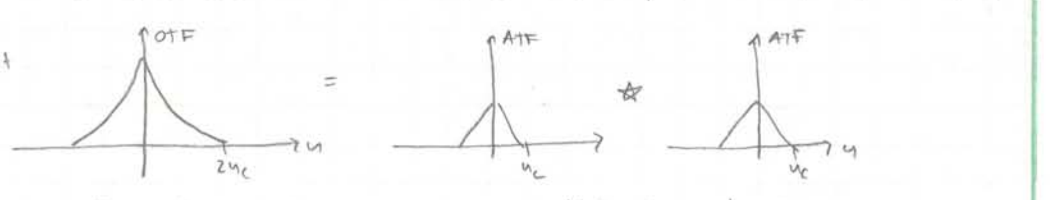
=



c) $MTF = |OTF| = |ATF \star ATF|$ ← incoherent transfer function, OTF is equal to autocorrelation of coherent transfer function, ATF

↑ auto correlation

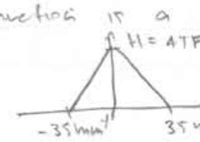
note that



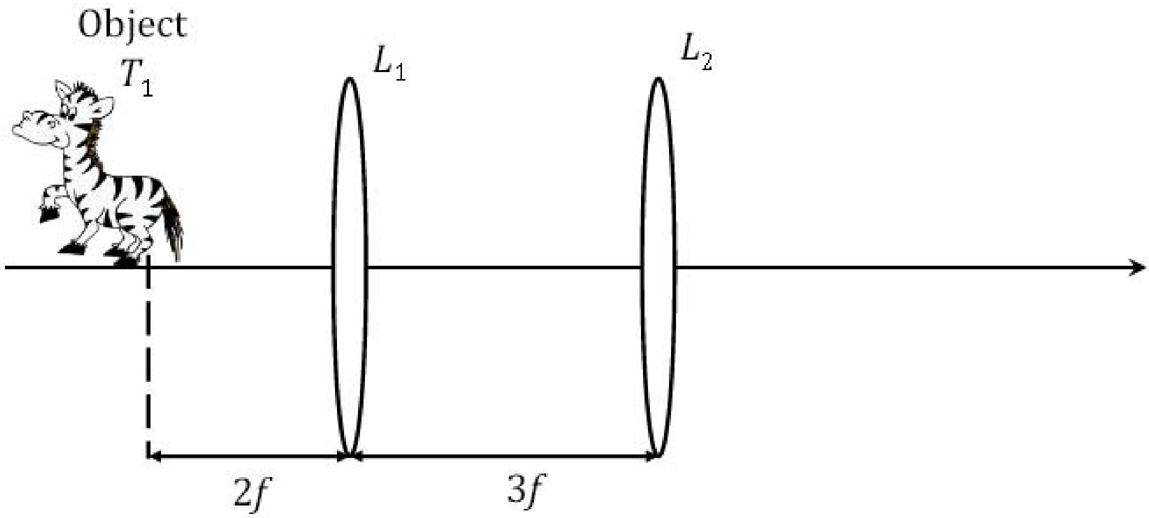
the cutoff freq for coherent illumination is just 1/2 half of the maximum allowed freq by the OTF (or MTF). since plot of MTF ends @ $u = 70\,mm^{-1}$ take that to be $2u_c$

$$\boxed{u_c = 70\,mm^{-1}/2 = 35\,mm^{-1}}$$

and coherent transfer function is a triangle with 1/2 width of $35\,mm^{-1}$



4. Consider the optical system shown the following schematic, where lenses L1, L2 are identical with focal length f and diameter 2a. A thin-transparency object T1 is placed at distance 2f to the left of L1.



a) **Where is the image formed? Use geometrical optics, ignoring the lens apertures for the moment.**

From the lens formula, we can calculate the location of the image after L1 and L2 as:

$$\frac{1}{2f}+\frac{1}{s_{i1}}=\frac{1}{f}, \ \therefore s_{i1}=2f$$
$$s_{o2}=3f-s_{i1}=f$$
$$\frac{1}{f}+\frac{1}{s_{i2}}=\frac{1}{f}, \ \therefore s_{i2}=\infty$$

Therefore, the image is located at infinity, to the right of L2.

b) **If the object T1 is an on-axis point source, describe the Fraunhofer diffraction pattern of the field to the right of L2.**

The object T1 is imaged to the front focal plane of L2, and therefore is turned into a plane wave after passing through L2. The Fraunhofer diffraction pattern a uniform plane wave.

c) **How are your two previous answers consistent within the approximations of paraxial geometrical and wave optics?**

Answer for (a) obtained with geometrical ray optics matches the answer from (b), which is obtained from wave optics calculation under the paraxial approximation. Both predicts that the image after L2 will propagate straight to infinity, in the form of a plane wave.

d) **The point source object T1 is replaced by a clear aperture of full width w and a second thin transparency T2 is placed between the two lenses, at distance f to**

the left of L2. The system is illuminated coherently with a monochromatic on-axis plane wave at wavelength λ. Write an expression for the field at distance 2f to the right of L2 and interpret the expression that you found.

A monochromatic on-axis plane wave hits the aperture of full width w at T1. This plane wave focuses into a point at the focal plane of L1, and is imaged as a point at a distance 2f away from L2. With no waveplates, a plane wave focuses into a point, at a distance 2f to the right of L2.

Let us now consider the effect of transparencies. The 2D problem in (x,z) is calculated. The field on the T1 plane can be expressed as:

$$T_1(x_1) = 1 \cdot rect(\frac{x_1}{W}) \ .$$

The plane wave focuses at the distance f to the right of L1.

Since T2 is located exactly at the image plane of T1, the image of T1 is multiplied to the transparency T2.

$$T_2'(x_2) = rect(\frac{-x_2}{W})\mathrm{T}_2(x_2)$$

Effectively, this modified transparency is illuminated with a point source located 2f to the left of L2. At the plane of T2, this illumination can be expressed as:

$$E_-(x_2) = \frac{1}{j\lambda f} e^{jkf} e^{jk\frac{x_2^2}{2f}}$$

$$E_+(x_2) = E_-(x_2)T_2'(x_2) = \frac{1}{j\lambda f} e^{jkf} e^{jk\frac{x_2^2}{2f}} rect(\frac{-x_2}{W})\mathrm{T}_2(x_2)$$

Because T2 is located at the focus of L2, the Fourier transform of E_{T_2+} is located at the distance f to the right of L2.

$$E_{L_2+f}(x) = \frac{We^{jkf}}{j\lambda f} j\sqrt{\lambda f} \exp[-j\pi\lambda f \cdot x^2]\mathrm{sinc}(-W\frac{x}{\lambda f})\mathrm{FT}\{\mathrm{T}_2\}|_{\frac{x}{\lambda f}}$$

At the distance 2f to the right of L2, this field is Fresnel propagated for an additional distance f, so the analytical expression can be written as:

$$E_{L_2+2f}(X) = \frac{\mathrm{e}^{jkf}\mathrm{e}^{jk\frac{X^2}{f}}}{j\lambda f} \int_{-\infty}^{\infty} E_{L_2+f}(x)\exp[-j2\pi x\frac{X}{\lambda f}]\exp[jk\frac{x^2}{f}]dx \ .$$

Qualitatively, this image resembles a point, if T2 does not produce additional spatial frequency components.

e) Derive and sketch approximately, with as much quantitative detail as you can, the intensity observed at distance 2f to the right of L2 when T2 is an infinite sinusoidal amplitude grating of period Λ, such that Λ « a.

With T2 being an infinite sinusoidal amplitude grating,

$$T_2(x_2) = \frac{1}{2}[1+\cos\left(\frac{2\pi x_2}{\Lambda}\right)] = \frac{1}{2}+\frac{1}{4}e^{i2\pi x_2/\Lambda}+\frac{1}{4}e^{-i2\pi x_2/\Lambda} .$$

The field after passing through T2 can now be written as:

$$E_+(x_2) = \frac{1}{j\lambda f}e^{jkf}e^{jk\frac{x_2^2}{2f}}rect(\frac{-x_2}{W})\left(\frac{1}{2}+\frac{1}{4}e^{i2\pi x_2/\Lambda}+\frac{1}{4}e^{-i2\pi x_2/\Lambda}\right).$$

At the focal plane of L2, the field can be calculated as:

$$E_{L_2+f}(x) = \frac{We^{jkf}}{\sqrt{\lambda f}}\exp[-j\pi\lambda f\cdot x^2]\mathrm{sinc}(-W\frac{x}{\lambda f})\left(\frac{1}{2}\delta(\frac{x}{\lambda f})+\frac{1}{4}\delta(\frac{x}{\lambda f}-\frac{1}{\Lambda})+\frac{1}{4}\delta(\frac{x}{\lambda f}+\frac{1}{\Lambda})\right)$$
$$= \frac{We^{jkf}}{4\sqrt{\lambda f}}\left[2\delta(\frac{x}{\lambda f})+\mathrm{sinc}(\frac{W}{\Lambda})\mathrm{e}^{-j\pi\frac{\lambda^3 f^3}{\Lambda^2}}\left(\delta(\frac{x}{\lambda f}-\frac{1}{\Lambda})+\delta(\frac{x}{\lambda f}+\frac{1}{\Lambda})\right)\right]$$

This represents three delta function sources at the focal plane of L2. At the plane 2f to the right of L2, the field can be analytically expressed as:

$$E_{L_2+2f}(X) = \frac{\mathrm{e}^{jkf}\mathrm{e}^{jk\frac{X^2}{f}}}{j\lambda f}\int_{-\infty}^{\infty}\frac{We^{jkf}}{4\sqrt{\lambda f}}\left[2\delta(\frac{x}{\lambda f})+\mathrm{sinc}(\frac{W}{\Lambda})\mathrm{e}^{-j\pi\frac{\lambda^3 f^3}{\Lambda^2}}\left(\delta(\frac{x}{\lambda f}-\frac{1}{\Lambda})+\delta(\frac{x}{\lambda f}+\frac{1}{\Lambda})\right)\right]\ldots$$
$$\ldots\exp[-j2\pi x\frac{X}{\lambda f}]\exp[jk\frac{x^2}{f}]dx$$

$$= \frac{\mathrm{e}^{2jkf}\mathrm{e}^{jk\frac{X^2}{f}}}{4j\lambda^{3/2}f^{3/2}}\left[2+\mathrm{sinc}(\frac{W}{\Lambda})\mathrm{e}^{-j\pi\frac{\lambda^3 f^3}{\Lambda^2}}\exp[j2\pi\frac{\lambda f}{\Lambda}]\left(\exp[-j2\pi\frac{X}{\Lambda}]+\exp[j2\pi\frac{X}{\Lambda}]\right)\right]$$
$$= C\,\mathrm{e}^{jk\frac{X^2}{f}}\left[1+\mathrm{e}^{-j\pi\frac{\lambda^3 f^3}{\Lambda^2}+j2\pi\frac{\lambda f}{\Lambda}}\mathrm{sinc}(\frac{W}{\Lambda})\sin(2\pi\frac{X}{\Lambda})\right]$$

This is a sinusoidal interference pattern with the periodicity X= Λ, multiplied with a quadratic phase factor.

Problem 5: Zernicke phase mask For problem 1 system are presented here. As shown in Fig. A in problem 1, x, x'', and x' are the lateral coordinates at the input, Fourier, and output plane, respectively. The complex transparencies at the input and Fourier plane are denoted by $t_1(x)$ and $t_2(x'')$, respectively. With on–axis plane illumination, we can formulate as follows:

1. field immediately after T1: $t_1(x)$
2. field immediately before T2: $\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x''}{\lambda f_1}}$
3. field immediately after T2: $t_2(x'')\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x''}{\lambda f_1}}$
4. field at the image plane: $\mathfrak{F}\left[t_2(x'')\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x''}{\lambda f_1}}\right]_{x''\rightarrow\frac{x'}{\lambda f_2}}$

$$= \mathfrak{F}\left[t_2(x'')\right]_{x''\rightarrow\frac{x'}{\lambda f_2}} \otimes \mathfrak{F}\left[\mathfrak{F}\left[t_1(x)\right]_{x\rightarrow\frac{x'}{\lambda f_1}}\right]_{x''\rightarrow\frac{x'}{\lambda f_2}} = \mathfrak{F}\left[t_2(x'')\right]_{x''\rightarrow\frac{x'}{\lambda f_2}} \otimes t_1\left(-\frac{f_2}{f_1}x'\right), \tag{1}$$

where we use $\mathfrak{F}\left[\mathfrak{F}\left[g(x)\right]\right] = g(-x)$. Note that the field at the image plane is a convolution of the scaled object field and the Fourier transform of the pupil function, where the FT of the pupil is the point spread function of the system.

Next, it is important to model correctly the transparencies of the gratings. For T_1, the phase delay caused by grooves is $\frac{2\pi}{\lambda}(n-1)d_1$, where d_1 is the height of the groove (1 μm), and the phase profile is shown in Fig. 1. Hence, the complex transparency of

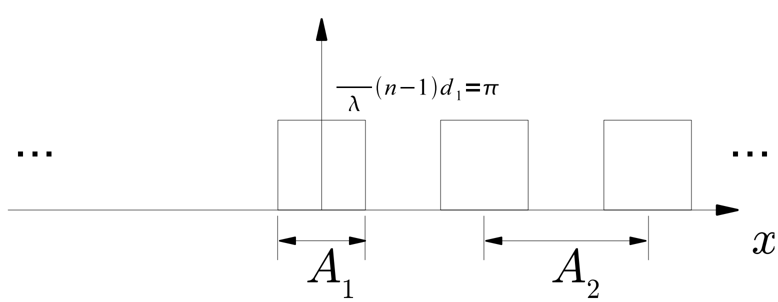


Figure 1: phase profile of the grating $T_1(x)$

T_1 is written as

$$t_1(x) = e^{i\phi_1(x)} = \exp\left\{ i\frac{2\pi}{\lambda}(n-1)d_1 \;\; \mathrm{rect} \;\; \frac{x}{A_1} \;\; \otimes \mathrm{comb} \;\; \frac{x}{A_2}\Big)\Big]\right\}, \tag{2}$$

where $A_1 = 5\ \mu$m and $A_2 = 10\ \mu$m. Hence,

$$t_1(x) = \begin{cases} e^{i\pi}(=-1) & \text{if } |x| < A_1/2, \\ \quad 1 & \text{if } A_1/2 < x < A_2/2 \text{ or } -A_2/2 < x < -A_1/2, \end{cases} \tag{3}$$

for $|x| < A_2/2$. Using the Fourier series ($\because$ $t_1(x)$ is periodic) and $A = A_2 = 2A_1$, we find the Fourier series coefficients as

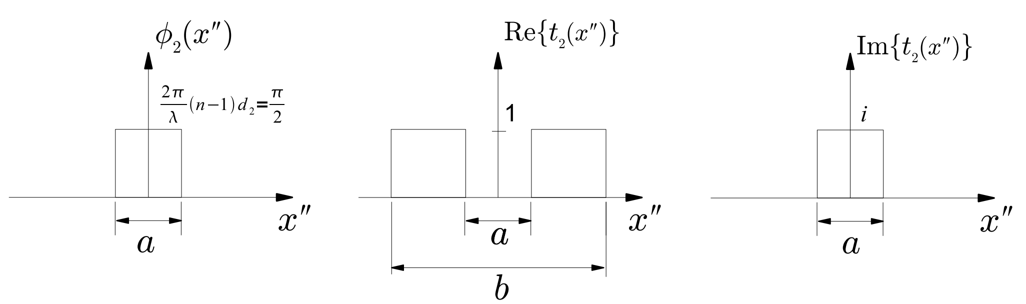
$$\begin{aligned} c_q &= \frac{1}{A}\int_{-A/2}^{A/2} t_1(x) e^{-i\frac{2\pi}{A}qx}\mathrm{d}x \\ &= \frac{1}{A}\left[\int_{-A/2}^{-A/4} e^{-i\frac{2\pi}{A}qx}\mathrm{d}x - \int_{-A/4}^{A/4} e^{-i\frac{2\pi}{A}qx}\mathrm{d}x + \int_{A/4}^{A/2} e^{-i\frac{2\pi}{A}qx}\mathrm{d}x\right] \end{aligned} \tag{4}$$

For $q = 0$, $c_0 = \frac{1}{A}\int t_1(x)\mathrm{d}x = 0$.
For $q \neq 0$,

$$\begin{aligned} c_q &= \frac{1}{A}\left[\left.\frac{e^{-i\frac{2\pi}{A}qx}}{-i\frac{2\pi}{A}q}\right|_{-A/2}^{-A/4} - \left.\frac{e^{-i\frac{2\pi}{A}qx}}{-i\frac{2\pi}{A}q}\right|_{-A/4}^{A/4} + \left.\frac{e^{-i\frac{2\pi}{A}qx}}{-i\frac{2\pi}{A}q}\right|_{A/4}^{A/2}\right] \\ &= \frac{1}{A}\;\; \frac{e^{i\frac{\pi}{2}q} - e^{i\pi q}}{-i\frac{2\pi}{A}q} - \frac{e^{-i\frac{\pi}{2}q} - e^{i\frac{\pi}{2}q}}{-i\frac{2\pi}{A}q} + \frac{e^{-i\pi q} - e^{-i\frac{\pi}{2}q}}{-i\frac{2\pi}{A}q} \\ &= \frac{e^{i\pi q} - e^{-i\pi q}}{i2\pi q} - \frac{e^{i\frac{\pi}{2}q} - e^{-i\frac{\pi}{2}q}}{i2\frac{\pi}{2}q} = \mathrm{sinc}\,(q) - \mathrm{sinc}\left(\frac{q}{2}\right) = -\mathrm{sinc}\left(\frac{q}{2}\right). \end{aligned}$$

Thus, $c_q = -\mathrm{sinc}\left(\frac{q}{2}\right) + \delta(q)$; all even orders disappear and only odd orders survive.

For the grating T_2, the phase profile is shown in Fig. 2.



(a) phase profile (b) real part (c) real part

Figure 2: complex transparency of the grating $T_2(x'')$

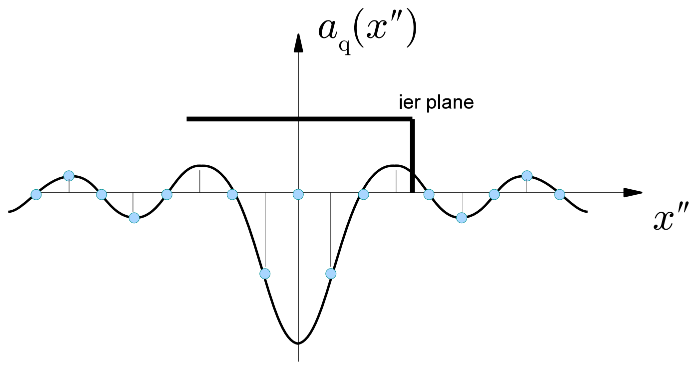


Figure 3: the field immediately before T_2.

The complex transparency can be written as

$$t_2(x'') = \left\{ \mathrm{rect}\left(\frac{x''}{b}\right) - \mathrm{rect}\left(\frac{x''}{a}\right)\right\} + i\mathrm{rect}\left(\frac{x''}{a}\right). \tag{5}$$

a) the intensity immediately after T1 is 1 because $|t_1(x)|^2 = 1$. Since T1 is a pure phase object and there is no intensity variation.

b) the field immediately before T2 can be computed from the Fourier series coefficients of $t_1(x)$. Since the period of T_1 is A, the diffraction angle of the order q is $\theta_q = q\frac{\lambda}{A}$, and the diffraction order q is focused at $f_1\theta_q$ on the Fourier plane. Hence, the field immediately before T_2 is

$$\sum_{q=-\infty}^{\infty} (\delta(q) - \mathrm{sinc}\,(q/2))\,\delta\left(x'' - q\frac{f_1\lambda}{A}\right) = \sum_{q=-\infty}^{\infty} (\delta(q) - \mathrm{sinc}\,(q/2))\,\delta\left(x'' - q \text{ cm}\right). \tag{6}$$

c) Since b (the width of the grating T_2) is 7 cm, the diffraction orders passing through the grating T_2 are $q = -3, -1, +1, +3$, where -1 and $+1$ orders get phase delay of $\pi/2$. The field immediately after the grating is

$$-\mathrm{sinc}\left(\frac{3}{2}\right)[\delta(x''-3) + \delta(x''+3)] - e^{i\frac{\pi}{2}}\mathrm{sinc}\left(\frac{1}{2}\right)[\delta(x''-1) + \delta(x''+1)]. \tag{7}$$

The field at the image plane is the Fourier transform of the field immediately after the grating T_2, which is computed as

$$\mathfrak{F}\left\{-\mathrm{sinc}\left(\frac{3}{2}\right)[\delta(x''-3) + \delta(x''+3)] - i\mathrm{sinc}\left(\frac{1}{2}\right)[\delta(x''-1) + \delta(x''+1)]\right\}_{u''\to\frac{x'}{\lambda f}} =$$

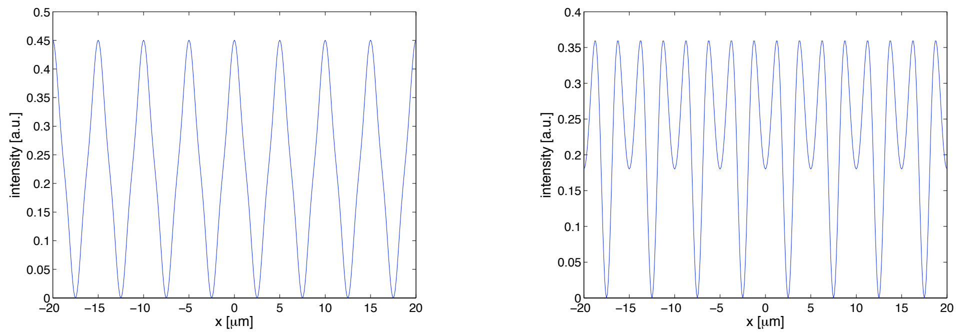
$$-2\mathrm{sinc}\left(\frac{3}{2}\right)\mathfrak{F}\left\{\frac{\delta(x''-q) + \delta(x''+q)}{2}\right\} - 2i\mathrm{sinc}\left(\frac{1}{2}\right)\mathfrak{F}\left\{\frac{\delta(x''-1) + \delta(x''+1)}{2}\right\} =$$

$$-2\mathrm{sinc}\left(\frac{3}{2}\right)\cos(2\pi 3u) - 2i\mathrm{sinc}\left(\frac{1}{2}\right)\cos(2\pi u) = -2\frac{-2}{3\pi}\cos\left(2\pi\frac{3x}{\lambda f_2}\right) - 2i\frac{2}{\pi}\cos\left(2\pi\frac{x}{\lambda f_2}\right) =$$

$$\frac{4}{3\pi}\cos\left(\frac{2\pi}{\lambda}(0.3)x\right) - i\frac{4}{\pi}\cos\left(\frac{2\pi}{\lambda}(0.1)x\right). \tag{8}$$

The intensity at the image plane is

$$I(x) \sim \left| \frac{1}{3} \cos\left(\frac{2\pi}{\lambda}(0.3)x \right) - i \cos\left(\frac{2\pi}{\lambda}(0.1)x \right) \right|^2 = \frac{1}{9} \cos^2\left(\frac{2\pi}{\lambda}(0.3)x \right) + \cos^2\left(\frac{2\pi}{\lambda}(0.1)x \right) \quad (9)$$



(a) with the phase mask

(b) with the phase mask

Figure 4: intensity pattern at the image plane

Figure 4 shows the intensity pattern at the image plane with and without the phase mask.

d) In Fig. 4, the phase mask introduces more dramatic intensity contrast, whose frequency is proportional to the twice of the spatial frequency of the object grating. In Fig. 4(b), there is a intensity variation but the contrast is smaller. This phase mask is particularly useful for imaging phase object because phase variation is converted into intensity variation.

e) In Fig. 4(a), although all the orders are recovered, the field signal is not identical as the input field (the field immediately after T1). Hence, we may still able to observe some intensity variation although the contrast could be very limited (but still better than the case without the phase mask).

f) If $a = 0\ .5$ cm, then the first order does not get the phase delay, and all the orders are imaged at the image plane. The output field is identical to the input field (the field immediately after T1); no intensity variation is produced. Intuitively, in Fig. 4(b), as all the order contribute, the valleys of the intensity pattern is filled and eventually uniform intensity pattern is produced.

Another solution with an alternative definition of the grating T1

(a) T1 is a phase mask; therefore, under the scalar and paraxial approximations, T1 does not modify the incident intensity

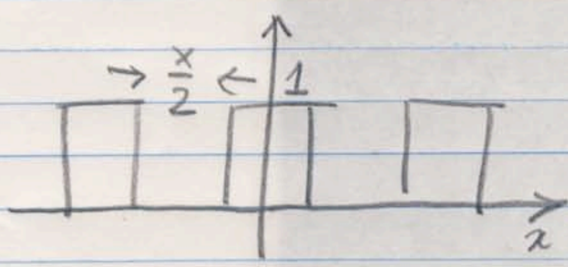
$$I(x)\Big|_{\substack{\text{just}\\\text{after}\\T_1}} = \left| e^{i\phi(x)} \right|^2 = 1 \quad \text{(uniform)}$$

where

$$\underset{\substack{\uparrow\\\text{one}\\\text{period}}}{\phi(x)} = \begin{cases} 0 & \text{if } |x| < 2.5\,\mu m \\ \frac{2\pi}{\lambda}(n-1)\,t = \frac{2\pi}{1\,\mu m}(1.5-1)\times 1\,\mu m = \pi & \text{if } -5\,\mu m < x < -2.5\,\mu m \text{ or } 2.5\,\mu m < x < 5\,\mu m \end{cases}$$

(periodically repeating with period = 10 μm)

(b). Just before T2, the optical field is the Fourier transform of T1, scaled by λf_1. Using the formula [Goodman p. 126, Figure 5.21]



$$= \sum_{n=-\infty}^{\infty} \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}\, e^{i\frac{2\pi n x}{X}} \equiv a(x)$$

we obtain

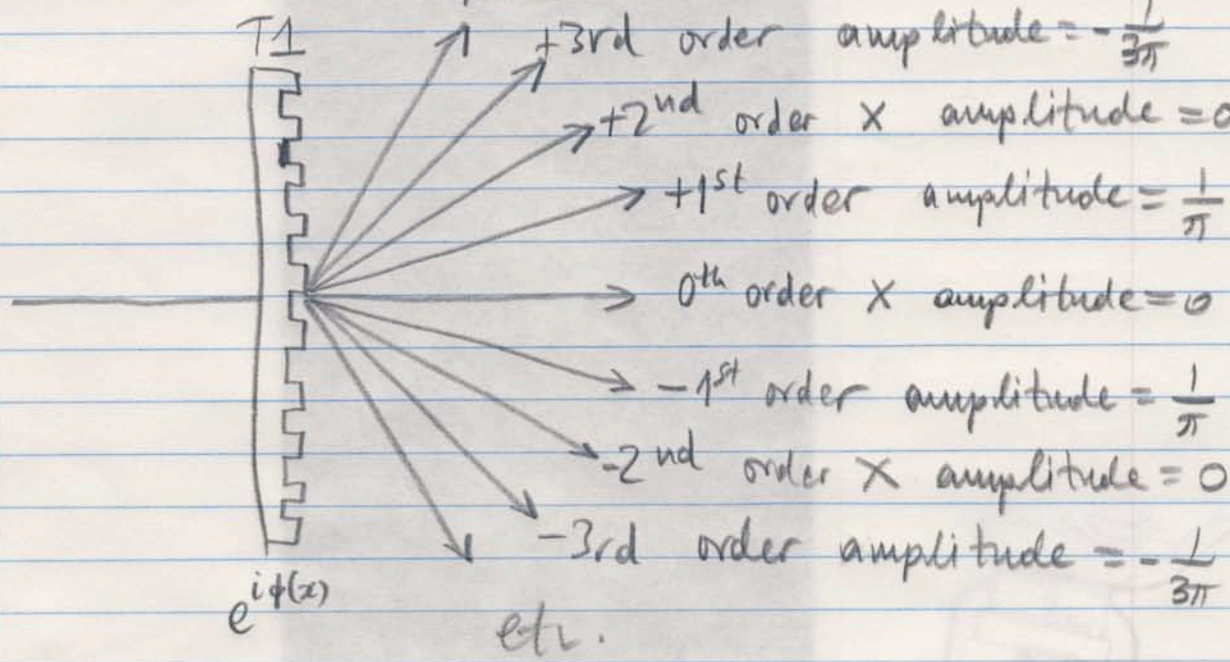
$$e^{i\phi(x)} = 2\left[a(x) - \frac{1}{2}\right] = \sum_{\substack{n=-\infty\\n\neq 0}}^{\infty} \frac{\sin\left(\frac{\pi n}{2}\right)}{\pi n}\, e^{i\frac{2\pi n x}{X}}$$

where $X = 10\,\mu m$

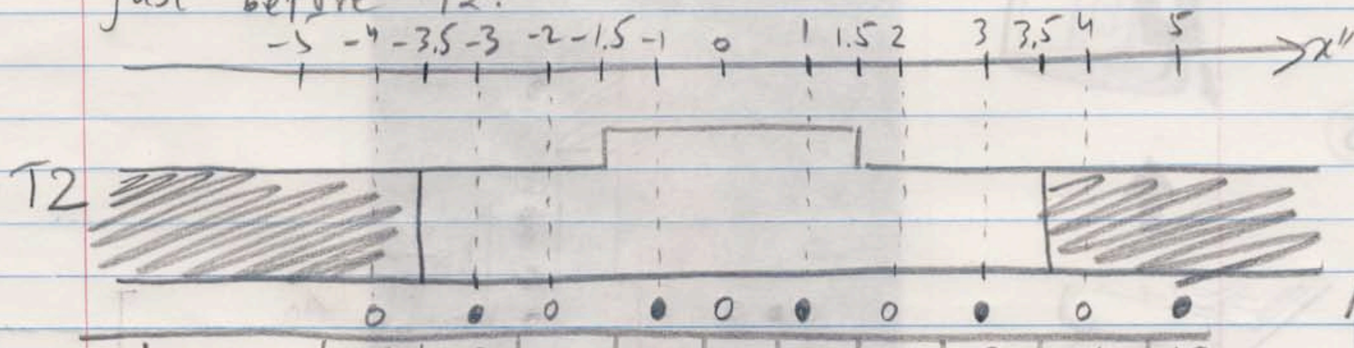
Note $\frac{\sin(\frac{n\pi}{2})}{n\pi}$ values

n	... -5	-4	-3	-2	-1	0	1	2	3 ...
$\frac{\sin(\frac{n\pi}{2})}{n\pi}$	$+\frac{1}{5\pi}$	0	$-\frac{1}{3\pi}$	0	$+\frac{1}{\pi}$	1	$+\frac{1}{\pi}$	0	$-\frac{1}{3\pi}$

These are the <u>amplitudes</u> of the diffracted orders



Lens L1 focuses each diffracted order to a focal point just before T2:



order	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$	$+5$
amplitude	0	$-\frac{1}{3\pi}$	0	$\frac{1}{\pi}$	0	$\frac{1}{\pi}$	0	$-\frac{1}{3\pi}$	0	$\frac{1}{5\pi}$ etc.
mask effect	cut off	$1\times$	$1\times$	$e^{-j\frac{\pi}{2}}\times$	$e^{-j\frac{\pi}{2}}\times$	$e^{-j\frac{\pi}{2}}\times$	$1\times$	$1\times$	cut off	cut off

The spacing between diffracted orders is

$$\frac{\lambda f_1}{\left(\begin{smallmatrix}\text{grating}\\ \text{period}\end{smallmatrix}\right)} = \frac{1\,\mu m \times 10\,cm}{10\,\mu m} = 1\,cm$$

c) The phase shift applied by the elevated portion (center) of the T2 (pupil plane) mask is

$$\frac{2\pi}{\lambda}(n-1)\, t_{T2} = \frac{2\pi}{1\,\mu m}(1.5-1) \times 0.5\,\mu m = \frac{\pi}{2}$$

So, <u>before</u> mask T2 the field is

$$\frac{1}{2}\Big\{ \cdots \frac{1}{5\pi}\delta(x''+5) - \frac{1}{3\pi}\delta(x''+3) + \frac{1}{\pi}\delta(x''+1) +$$
$$+ \frac{1}{\pi}\delta(x''-1) - \frac{1}{3\pi}\delta(x''-3) + \frac{1}{5\pi}\delta(x''-5) + \cdots \Big\}$$

(c) The mask T2 cuts off all orders beyond the $\pm 4^{th}$, and phase shifts the ± 1st order with respect to the ± 3rd orders. So just <u>after</u> mask T2 the field is

$$\frac{1}{2}\Big\{ -\frac{1}{3\pi}\delta(x''+3) + \frac{e^{-i\frac{\pi}{2}}}{\pi}\delta(x''+1) + \frac{e^{-i\frac{\pi}{2}}}{\pi}\delta(x''-1) +$$
$$+ \left(-\frac{1}{3\pi}\right)\delta(x''+3) \Big\}$$

At the output plane, the field is the Fourier transform, of the field just after T2, i.e.

scaled by $\frac{1}{\lambda f_2}$ // recall $e^{-i\frac{\pi}{2}} = -i$ //

$$\frac{1}{2}\left\{-\frac{1}{3\pi}e^{-i2\pi\frac{3x'}{10\mu m}} - \frac{i}{\pi}e^{-i2\pi\frac{x'}{10\mu m}} - \frac{i}{\pi}e^{+i2\pi\frac{x'}{10\mu m}} - \frac{1}{3\pi}e^{+i2\pi\frac{x'}{10\mu m}}\right\} =$$

$$= -\frac{1}{3\pi}\cos\left(2\pi\frac{3x'}{10\mu m}\right) - \frac{i}{\pi}\cos\left(2\pi\frac{x'}{10\mu m}\right)$$

The intensity is:

$$I_{out}(x') = |\text{field}_{out}(x')|^2 =$$

$$= \frac{1}{9\pi^2}\cos^2\left(2\pi\frac{3x'}{10\mu m}\right) + \frac{1}{\pi^2}\cos^2\left(2\pi\frac{x'}{10\mu m}\right)$$

(d) (We recognize this as a Zernicke phase mask).
The original phase object T1 would have been invisible without the mask T2. With T2, we can observe intensity variation in the output plane.