1. Optical path length calculation using a thin lens
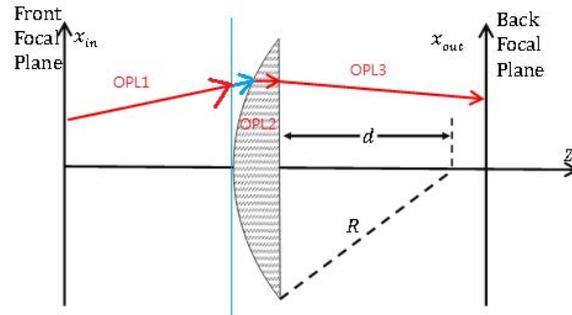

Figure 1.

a) The thin lens approximation can be applied to the lens maker's equation as:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2} + \frac{(n-1)d}{nR_1R_2}\right) \approx (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right). \quad (d << R)$$

Using $R_1 = R$, $R_2 = \infty$, we get: $\quad \frac{1}{f} = \frac{n-1}{R}, \quad f = \frac{R}{n-1}.$

$\therefore$ FFP is located $f = \dfrac{R}{n-1}$ to the left of the front surface of the lens.

$\therefore$ BFP is located $f = \dfrac{R}{n-1}$ to the right of the back surface of the lens.

Alternatively, the same results can be obtained from the ABCD matrix of the lens:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{n-1}{R} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}, \text{ which gives BFL} = \text{-A/C and FFL=-D/C.}$$

How much simplification is acceptable in using the paraxial approximation (small angle approximation)?

- **The effect of the paraxial approximation on the angle θ**

The small-angle approximation is a useful simplification of the basic trigonometric functions which is approximately true in the limit where the angle approaches zero. They are truncations of the Taylor series for the basic trigonometric functions to a second-order approximation.

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots \cong 1 - \frac{\theta^2}{2} + O(\theta^4)$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \cong \theta + O(\theta^3)$$

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots \cong \theta + O(\theta^3)$$

$$\sec\theta = \cos^{-1}\theta = 1 + \frac{\theta^2}{2} + \frac{5\theta^4}{24} + \cdots \cong 1 + \frac{\theta^2}{2} + O(\theta^4)$$

In all cases, the order of the error is limited to be above $O(\theta^3)$. Whether you choose to work with the original trigonometric functions or choose to work with alternative form such as $\sqrt{1+\theta^2}$, you can make sure that the linear and quadratic terms of θ are preserved.

$$\sqrt{1+\theta^2} = 1 + \frac{\theta^2}{2} - \frac{\theta^4}{8} + \cdots \cong 1 + \frac{\theta^2}{2} + O(x^4)$$

- **The effect of the paraxial approximation on the lateral displacement x**

In the paraxial approximation, the longitudinal and lateral dimensions can be also simplified using the fact that x is smaller than R and f.

$$\sqrt{R^2+x^2} = R\sqrt{1+\left(\frac{x}{R}\right)^2} = R\left[1 + \frac{\left(\frac{x}{R}\right)^2}{2} - \frac{\left(\frac{x}{R}\right)^4}{8} + \cdots\right] \cong R\left[1 + \frac{x^2}{2R^2} + O\left(\left(\frac{x}{R}\right)^4\right)\right]$$

$$\sqrt{f^2+x^2} \cong f\left[1 + \frac{x^2}{2f^2} + O\left(\left(\frac{x}{f}\right)^4\right)\right]$$

b) Calculate the OPL for an arbitrary paraxial ray.

Let us denote the path length inside air before the lens as OPL1, the path length inside the lens as OPL2, and the path after the lens as OPL3. For the three components, we know the distance of travel in z direction. For an arbitrary paraxial ray leaving x_{in} at an angle θ, we can write:

$$OPL_1 = f\sec\theta = \sqrt{f^2 + (f\tan\theta)^2} \cong f(1 + \tfrac{1}{2}\theta^2).$$

From the fact that the lens is thin, we can safely assume that the lateral height of the ray entering and leaving the lens stays the same. Also, we can neglect the small distance traveled in air in front of the lens, between OPL1 and OPL2, illustrated in the blue arrow in figure 1.

$$OPL_2 = n\cdot h(x,y) = n\sqrt{R^2 - (x_{in} + f\theta)^2} - nd \cong nR\left[1 - \frac{(x_{in} + f\theta)^2}{2R^2}\right] - nd \qquad \text{(using x<<R)}$$

Lastly, we can obtain the angle of the ray coming out of the lens to be $\theta' = \theta - \frac{x_{in} + f\theta}{f} = -\frac{x_{in}}{f}$. Therefore,

$$OPL_3 = \sqrt{f^2 + (f\theta')^2} = \sqrt{f^2 + x_{in}^2} \cong f\left[1 + \frac{x_{in}^2}{2f^2}\right].$$

Note that the OPL3 for all rays are identical, regardless of the angle. The wavefront plot in d) will clarify this fact further. Summing up the optical path lengths gives:

$$OPL_\theta \cong f(1 + \tfrac{1}{2}\theta^2) + nR\left[1 - \frac{(x_{in} + f\theta)^2}{2R^2}\right] - nd + f\left[1 + \frac{x_{in}^2}{2f^2}\right]$$

$$= 2f + n(R-d) + \frac{1}{2f}\left((f\theta)^2 + x_{in}^2 - \tfrac{n}{n-1}(x_{in} + f\theta)^2\right).$$

Note that the terms that are proportional to the quadratic order of θ are not crossed out.

c) For the chief ray, $f\theta = -x_{in}$, which gives

$$OPL_{chief} = 2f + n(R-d) + \frac{1}{2f}\left((f\theta)^2 + x_{in}^2 - \tfrac{n}{n-1}(0)^2\right) = 2f + n(R-d) + x_{in}^2/f.$$

Since a negative term is zero for a chief ray, it is longer than most other rays with

$$OPL_{chief} - OPL_\theta = x_{in}^2/f - \frac{1}{2f}\left((f\theta)^2 + x_{in}^2 - \tfrac{n}{n-1}(x_{in} + f\theta)^2\right)$$

$$= \frac{1}{2f}\left((x_{in}^2 - f^2\theta^2) + \tfrac{n}{n-1}(x_{in} + f\theta)^2\right) = \frac{1}{2f(n-1)}\left((2n-1)x_{in} + f\theta\right)(x_{in} + f\theta)$$

Setting x_{in} to be positive, the chief ray is longer for $x_{in} + f\theta > 0$, which corresponds to $-\frac{x_{in}}{f} < \theta$.

2.71 / 2.710 Optics
Spring 2014