2.341 Spring 2014

Problem 1 Solution:

(a) $V_\theta(r,Z) = r W(Z)$ $\qquad V_r = V_Z = 0$

$$\underline{\underline{\nabla V}} = \begin{bmatrix} \frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_Z}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r} & \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r} & \frac{1}{r}\frac{\partial V_Z}{\partial \theta} \\ \frac{\partial V_r}{\partial Z} & \frac{\partial V_\theta}{\partial Z} & \frac{\partial V_Z}{\partial Z} \end{bmatrix}$$

$$\Rightarrow \underline{\underline{\nabla V}} = \begin{bmatrix} 0 & W(Z) & 0 \\ -W(Z) & 0 & 0 \\ 0 & r\frac{dW}{dZ} & 0 \end{bmatrix}$$

$$\underline{\underline{\dot\gamma}} = \underline{\underline{\nabla V}} + (\underline{\underline{\nabla V}})^T \Rightarrow \underline{\underline{\dot\gamma}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r\frac{dW}{dZ} \\ 0 & r\frac{dW}{dZ} & 0 \end{bmatrix}$$

$$\boxed{\dot\gamma_{\theta Z} = \dot\gamma_{Z\theta} = r\frac{dW}{dZ}}$$

$$\begin{cases} I = tr(\underline{\underline{\dot\gamma}}) = 0 \\ II = \dot\gamma_{ij}\dot\gamma_{ji} = 0 + \dot\gamma_{\theta Z}\dot\gamma_{Z\theta} + \dot\gamma_{Z\theta}\dot\gamma_{\theta Z} = 2r^2\left(\frac{dW}{dZ}\right)^2 \\ III = \dot\gamma_{ij}\dot\gamma_{jK}\dot\gamma_{Ki} = 0 \end{cases}$$

But not a homogenous Shear flow ($\dot\gamma \neq$ Const.)

Yes this is a "Shear Flow".
Each disc at Z = Const. is a shearing Suface.
They move isometrically. and the distance between neighboring points remains Const.

(b)

Inertia terms $\sim \rho \dfrac{V^2}{L}$ $\quad V \sim R\Omega$

$L \sim R$

Viscous terms $\sim \dfrac{\mu V}{H^2}$ $\quad V \sim R\Omega$

$$\Rightarrow \frac{\text{Inertia}}{\text{Viscous}} \sim \frac{\rho V H}{\mu} \cdot \frac{H}{L} = \frac{\rho (R\Omega) H}{\mu} \cdot \frac{H}{R}$$

$$= \frac{\rho \Omega H^2}{\mu} \ll 1$$

Lubrication also needs $\dfrac{H}{R} \ll 1$

θ-Comp.:

Inertialess:

$$0 = -\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}\right] - \frac{1}{r}\frac{\partial P}{\partial \theta} + \rho g_\theta$$

$\rightarrow$ Sym. with $\theta \Rightarrow \dfrac{\partial P}{\partial \theta} = 0$, g_θ is also $= 0$, $\dfrac{\partial}{\partial \theta} = 0$

also by Knowing $\dot{\gamma}_{r\theta} = 0$, $\dot{\gamma}_{\theta r} = 0$ $\rightarrow$ the corresponding stress components will be zero $\Rightarrow \tau_{r\theta} = \tau_{\theta r} = 0$

Even if you think that $\tau_{r\theta}$ may be still non-zero by lubrication:

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \tau_{r\theta} \sim \frac{\tau}{r} \ll \frac{\tau}{H} \sim \frac{\partial}{\partial z}\tau_{z\theta} \Rightarrow \text{All the terms other than } \tau_{z\theta} \text{ will vanish} \ldots$$

$$\Rightarrow \frac{\partial}{\partial z}\tau_{z\theta} = 0 \Rightarrow \boxed{\tau_{z\theta} = \text{Const. with } z}$$

$$\tau_{Z\theta} = \text{fun}(\dot{\gamma}_{Z\theta})$$

$$\tau_{Z\theta} = \text{Const. with } Z \Rightarrow \dot{\gamma}_{Z\theta} = \text{Const. with } Z \Rightarrow r\frac{dW}{dZ} = C$$

$$\Rightarrow rW(Z) = C_1 Z + C_2$$

No slip boundary condition

$$Z=0,\ V_\theta = 0 \Rightarrow rW(Z) = 0 \Rightarrow C_2 = 0$$

$$Z=H,\ V_\theta = r\Omega \Rightarrow rW(Z) = r\Omega = C_1 H$$

$$\Rightarrow C_1 = \frac{r\Omega}{H}$$

$$\Rightarrow \boxed{V_\theta(r,Z) = rW(Z) = \frac{r\Omega}{H}Z} \Rightarrow W(Z) = \frac{\Omega}{H}Z$$

$$\dot{\gamma}_{\theta Z} = r\frac{dW}{dZ} = r\frac{\Omega}{H} \Rightarrow \boxed{\dot{\gamma}_{\theta Z}(r) = \frac{r\Omega}{H}}$$ changes linearly with r

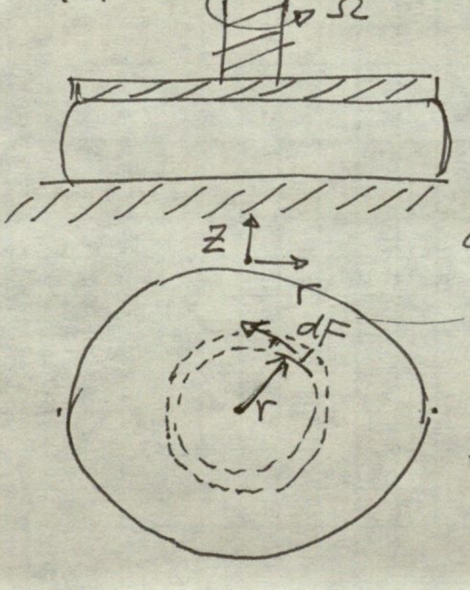
$$\Rightarrow \dot{\gamma}_{\theta Z}(R) = \frac{R\Omega}{H}$$

$$\Rightarrow \boxed{\dot{\gamma}(r) = \frac{r}{R}\dot{\gamma}_R}$$

(C) $T = 2\pi\int_0^R \eta(\dot{\gamma})\,\dot{\gamma}\,r^2 dr$??

Side View

Top View



$$dT = r\,\tau\,dA$$

$$dF = \tau\,dA$$

$$\tau = |\tau_{Z\theta}| = \eta\,\dot{\gamma}_{\theta Z}$$

$$dA = (2\pi r)\,dr$$

$$\Rightarrow T = \int dT = \int r\,dF = \int r\,\tau\,dA = \int_0^R r\,\eta(\dot\gamma_{\theta z})\,\dot\gamma_{\theta z}\,(2\pi r)\,dr$$

$$\Rightarrow \boxed{T = \int_0^R (2\pi) r^2\,\eta(\dot\gamma)\dot\gamma\,dr}$$

It is not possible to explicitely calculate this cause we do not know how ~~$\dot\gamma$~~ $\eta(\dot\gamma)$ changes with $\dot\gamma$ and consequently r...

$$\left.\begin{aligned} \dot\gamma_{(r)} = \frac{r}{R}\dot\gamma_R \Rightarrow dr &= \frac{R}{\dot\gamma_R}\,d\dot\gamma \\ r &= \frac{R\dot\gamma}{\dot\gamma_R} \end{aligned}\right\} \Rightarrow T = \int_0^{\dot\gamma_R} (2\pi)\,\frac{R^2}{\dot\gamma_R^2}\,\dot\gamma^2\,\eta(\dot\gamma)\,\dot\gamma\,\frac{R}{\dot\gamma_R}\,d\dot\gamma$$

$$= 2\pi R^3 \int_0^{\dot\gamma_R} \frac{\dot\gamma^3}{(\dot\gamma_R)^3}\,\eta(\dot\gamma)\,d\dot\gamma$$

$$\boxed{\Rightarrow T = \left(\frac{2\pi R^3}{\dot\gamma_R^3}\right) \int_0^{\dot\gamma_R} \eta(\dot\gamma)\dot\gamma^3\,d\dot\gamma}$$

(d) $$\Rightarrow \frac{T\dot\gamma_R^3}{2\pi R^3} = \int_0^{\dot\gamma_R} \eta(\dot\gamma)\dot\gamma^3\,d\dot\gamma$$

$$\Rightarrow \frac{d(T\dot\gamma_R^3/2\pi R^3)}{d\dot\gamma_R} = \eta(\dot\gamma_R)\dot\gamma_R^3 \Rightarrow \frac{d(T/2\pi R^3)}{d\dot\gamma_R}\dot\gamma_R^3 + (T/2\pi R^3)\,3\dot\gamma_R^2 = \eta(\dot\gamma_R)\dot\gamma_R^3$$

Leibniz rule

$$\Rightarrow \eta(\dot\gamma_R) = \frac{d(T/2\pi R^3)}{d\dot\gamma_R} + 3\frac{(T/2\pi R^3)}{\dot\gamma_R}$$

$$\Rightarrow \eta(\dot\gamma_R) = \frac{(T/2\pi R^3)}{\dot\gamma_R}\left(3 + \frac{d(T/2\pi R^3)/(T/2\pi R^3)}{d\dot\gamma_R/\dot\gamma_R}\right)$$

$$\Rightarrow \boxed{\eta(\dot{\gamma}_R) = \frac{(T/2\pi R^3)}{\dot{\gamma}_R}\left(3 + \frac{d \ln(T/2\pi R^3)}{d \ln \dot{\gamma}_R}\right)}$$

$$\Rightarrow \boxed{\begin{matrix} A = 2\pi R^3 \\ B = 3 \end{matrix}}$$

(e) Newtonian: $\eta(\dot{\gamma}) = \mu \Rightarrow T = \int_0^R (2\pi) r^2 \mu \dot{\gamma}\, dr$ $\quad \dot{\gamma} = \frac{r\Omega}{H}$

$$\Rightarrow T = 2\pi \int_0^R \mu \frac{\Omega}{H} r^3 dr = 2\pi \frac{\mu\Omega}{H}\frac{R^4}{4}$$

$$\Rightarrow \boxed{T = \frac{2\pi\mu\Omega R^4}{4H}}$$

$$T/2\pi R^3 = \frac{\mu\Omega R}{4H}$$

$$\dot{\gamma}_R = \frac{R\Omega}{H} \Rightarrow \frac{T/2\pi R^3}{\dot{\gamma}_R}\left(3 + \frac{d \ln(T/2\pi R^3)}{d \ln \dot{\gamma}_R}\right)$$

$$= \frac{\mu}{4}\left(3 + \frac{d\left[\ln\left(\frac{\mu}{4H}\right) + \ln \dot{\gamma}_R\right]}{d \ln \dot{\gamma}_R}\right)$$

$$= \frac{\mu}{4}(3+1) = \mu \checkmark$$

Power Law: $\eta(\dot{\gamma}) = K\dot{\gamma}^{n-1} \Rightarrow T = \int_0^R 2\pi r^2 K \dot{\gamma}^n dr$ $\quad \dot{\gamma} = \frac{r\Omega}{H}$

$$\Rightarrow T = \int_0^R (2\pi) K \left(\frac{\Omega}{H}\right)^n r^{n+2} dr = \frac{2\pi\Omega^n}{H^n} K \frac{R^{n+3}}{n+3}$$

$$\left.\begin{matrix} T/2\pi R^3 = \frac{\Omega^n}{H^n} K \frac{R^n}{n+3} \\ \dot{\gamma}_R = \frac{R\Omega}{H} \end{matrix}\right\} \Rightarrow$$ ... next page...

$$\cancel{d}\ \frac{T/(2\pi R^3)}{\dot\gamma_R}\left[3+\frac{d\ln(T/2\pi R^3)}{d\ln\dot\gamma_R}\right]$$

$$=\frac{\Omega^{n-1}}{H^{n-1}}K\frac{R^{n-1}}{n+3}\left[3+\frac{d\left[\ln(K/n+3)+n\ln(\dot\gamma_R)\right]}{d\ln\dot\gamma_R}\right]$$

$$=\frac{\Omega^{n-1}}{H^{n-1}}K\frac{R^{n-1}}{n+3}(3+n)=K\left(\frac{\Omega R}{H}\right)^{n-1}=K\dot\gamma_R^{\,n-1}=\eta(\dot\gamma_R)\ \checkmark\checkmark$$

(f) $$\frac{\eta(\dot\gamma)-\eta_s}{\eta_0-\eta_s}=\left[1+(\lambda\dot\gamma)^2\right]^{(n-1)/2}$$

$\eta_0=\eta(\dot\gamma=0)\longrightarrow\ \simeq 2\ Pa.s$

$\Rightarrow \boxed{\eta_0\simeq 2\ Pa.s}$

$\eta_s=\eta(\dot\gamma=\infty)\Rightarrow \boxed{\eta_s\simeq 2-3\ mPa.s \simeq 2\times10^{-3}\ Pa.s}$

$\lambda=\left(\text{Shear rate at which } \eta(\dot\gamma) \text{ falling starts}\right)^{-1}$

$=\left(10^0\ s^{-1}\right)^{-1}=1\,s\Rightarrow\boxed{\lambda\simeq 1\,sec}$

~~At~~ $\eta(\dot\gamma)\simeq\dot\gamma^{\,n-1}\Rightarrow$ n-1 power coefficient in $\eta(\dot\gamma)$ curve...

it drops for 10^0 Pa.s to 10^{-2} Pa.s as we go from $10^0\ s^{-1}$ to $10^3\ s^{-1}\Rightarrow n-1=\frac{-2}{3}\Rightarrow\boxed{n\simeq\frac{1}{3}}$

(g)

r-comp.

Inertia ignored:

$$\underline{\underline{\tau}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(\Psi_1+\Psi_2)\dot{\gamma}^2 & \eta\dot{\gamma} \\ 0 & \eta\dot{\gamma} & -\Psi_2\dot{\gamma}^2 \end{bmatrix}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) - \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} - \frac{\partial}{\partial z}\tau_{zr} + \frac{\tau_{\theta\theta}}{r} - \frac{\partial P}{\partial r} + \rho g_r = 0$$

$g_r = 0$; $\tau_{rr} = 0$; $\frac{\partial}{\partial \theta} = 0$; $\tau_{zr} = 0$;

$$\Rightarrow \frac{\tau_{\theta\theta}}{r} - \frac{\partial P}{\partial r} = 0 \Rightarrow \boxed{\frac{\partial P}{\partial r} = \frac{\tau_{\theta\theta}}{r}}$$

$$\Rightarrow \int_r^R \frac{\partial P}{\partial r} dr = \int_r^R \frac{\tau_{\theta\theta}}{r} dr \Rightarrow P(R) - P(r) = \int_r^R \frac{-(\Psi_1+\Psi_2)\dot{\gamma}^2}{r} dr$$

$$\Rightarrow P_a - P(r) = \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} \frac{-(\Psi_1+\Psi_2)\dot{\gamma}^2}{R\dot{\gamma}/\dot{\gamma}_R} \left(\frac{R}{\dot{\gamma}_R}\right) d\dot{\gamma}$$

$P(R) = P_a$

ignoring surface tension

$$\Rightarrow P(r) - P_a = \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} + (\Psi_1+\Psi_2)(\dot{\gamma})\, d\dot{\gamma}$$

" 10.2-20 in DPL Vol 1 "

(h) $\Pi_{zz}\, dA = dF \Rightarrow \left(P(r) + \cancel{\tau_{zz}}' - P_a\right) dA = dF$

$$\Rightarrow dF = \left(P(r) - P(a) + \tau_{zz}\right) dA$$

$$\Rightarrow dF\Big|_{r=0} = \left(P(r=0) - P_a + \Psi_2\dot{\gamma}^2(r=0)\right) dA$$

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$$= \left(\int_0^{\dot{\gamma}_R} + (\Psi_1+\Psi_2)\,\dot{\gamma}\, d\dot{\gamma}\right) dA \rightarrow \text{It is finite}$$

Normal Force on the plate: F

$$dF = \left(P(r) - P(a) + -\psi_2 \dot{\gamma}^2 \right) dA$$

$$P(r) - P(a) = \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} (\psi_1 + \psi_2)\,\dot{\gamma}\, d\dot{\gamma}$$

$$\Rightarrow F = \int_0^R \underbrace{(2\pi r\, dr)}_{dA} \left[-\psi_2 \dot{\gamma}^2 + \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} (\psi_1 + \psi_2)\,\dot{\gamma}\, d\dot{\gamma} \right]$$

Knowing that $r = \frac{R\dot{\gamma}}{\dot{\gamma}_R}$ we can change r into $\dot{\gamma}_r$ and the integral will be:

$$\boxed{F = \left((2\pi R^2)/\dot{\gamma}_R^2 \right) \int_0^{\dot{\gamma}_R} \left[-\psi_2 \dot{\gamma}^3 d\dot{\gamma}_r + \dot{\gamma}_r d\dot{\gamma}_r \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} (\psi_1 + \psi_2)\,\dot{\gamma}\, d\dot{\gamma} \right]} \quad (*)$$

Using integration by parts and the fact that

$$\int u\, dv = uv - \int v\, du$$

(here it's convenient to say $u = \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} (\psi_1 + \psi_2)\,\dot{\gamma}\, d\dot{\gamma}$

and $dV = \dot{\gamma}_r\, d\dot{\gamma}_r$

$$\Rightarrow V = \dot{\gamma}_r^2/2 \quad \text{and} \quad du = -\left(\psi_1(\dot{\gamma}_r) + \psi_2(\dot{\gamma}_r)\right)\dot{\gamma}_r$$

$$\Rightarrow \int_0^{\dot{\gamma}_R} \dot{\gamma}_r\, d\dot{\gamma}_r \int_{\dot{\gamma}_r}^{\dot{\gamma}_R} (\psi_1 + \psi_2)\,\dot{\gamma}\, d\dot{\gamma} = \int_0^{\dot{\gamma}_R} \frac{\dot{\gamma}_r^3}{2} (\psi_1 + \psi_2)\, d\dot{\gamma}_r$$

$\Rightarrow$ Plugging into (*) :
$$\boxed{F = \left(\frac{\pi R^2}{\dot{\gamma}_R^2} \right) \int_0^{\dot{\gamma}_R} \dot{\gamma}_r^3 (\psi_1 - \psi_2)\, d\dot{\gamma}_r}$$

Now similar to parts "c" and "d" you can rearrange and use Leibniz rule to show that:

$$\Psi_1(\dot{\gamma}_R) - \Psi_2(\dot{\gamma}_R) = \frac{1}{\dot{\gamma}_R^{\,2}} \left(\frac{F}{\pi R^2}\right) \left[2 + \frac{d \ln (F/\pi R^2)}{d \ln \dot{\gamma}_R} \right]$$

which is a "nice" expression. It helps us to get rheological measurements out of a "non-homogenous" shear flow for any arbitrary liquid with any possible constitutive equation!!

End of Solution

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